# Chapter 9

# dc-ac Inverters



## INTRODUCTION

In this chapter, we will consider power electronic circuits that produce variable-frequency ac output voltages from dc sources. As discussed in Chapter 3, depending on whether the source is dc or ac, power electronic circuits with ac output voltages are referred to as *dc-ac inverters* or *ac-ac cycloconverters*. In converting ac to ac, if the output voltage frequency is different from the source frequency, the converter is called an *ac voltage controller*. Traditionally, dc-to-ac inverters (also known as *static inverters*) use fixed dc sources to produce symmetrical ac output voltages at fixed or variable frequency or magnitude. The output ac voltage system can be of the single-phase or three-phase type at frequencies of 50 Hz, 60 Hz, and 400 Hz with a voltage magnitude range of 110–380 VAC. Inverter circuits are used to deliver power from a dc source to a passive or active ac load employing conventional SCRs or gate-driven semiconductor devices such as GTOs, IGBTs, and MOSFETs. Due to their increased switching speed and power capabilities coupled with complex control techniques, today's inverters can operate in wide ranges of regulated output voltage and frequency with reduced harmonics. Medium- and high-power half-bridge and full-bridge switching devices using MOSFETs and IGBTs are available as packages.

Dc-to-ac inverters are used in applications where the only source available is a fixed dc source and the system requires an ac load such as in uninterruptible power supply (UPS). Applications where dc-ac inverters are used include aircraft power supplies, variable-speed ac motor drives, and lagging or leading var generation. For example, an inverter used to provide necessary changes in the frequency of the ac output is used to regulate the speed of an induction motor and is also used in a UPS system to produce a fixed ac frequency output when the main power grid system is out.

## 9.1 BASIC BLOCK DIAGRAM OF DC-AC INVERTER

Figure 9.1 shows a typical block diagram of a power electronic circuit utilizing a dc-to-ac inverter with input and output filters used to smooth the output ac signal.

The feedback circuit is used to sense the output voltage and compare it with a sinusoidal reference signal as shown in Fig. 9.1. Depending on the arrangement of the power switches and their types, various control techniques are normally adopted in industry. The control objective is to produce a controllable ac output from an uncontrollable dc voltage source. Even though the desired output voltage waveform is purely sinusoidal, practical inverters are not purely sinusoidal but include significant high-frequency harmonics. This is why inverters normally employ a high-frequency switching technique to reduce such harmonics. Two alternative control methods of inversions will be discussed in this chapter: uniform pulse width modulation (PWM) and sinusoidal PWM. Depending on the output power level, both techniques find their way into practice. Sinusoidal PWM is widely used in motor drive applications with high-frequency operation.

The front end of the power electronic circuit in Fig. 9.1 is the line ac-to-dc converter discussed in Chapters 7 and 8. Unlike the line ac-to-dc converter circuits in which diodes and SCRs are commutated naturally, in dc-to-ac inverters forced commutation is used to turn on and off the power switches. The negative portion of the source voltage is used naturally to turn off the diode and/or the SRC in line ac-dc converters. As a result, the ac-dc converter circuit has one ac frequency—the line frequency—and the device relies on line commutation for switching. In dc-to-ac inverters, the

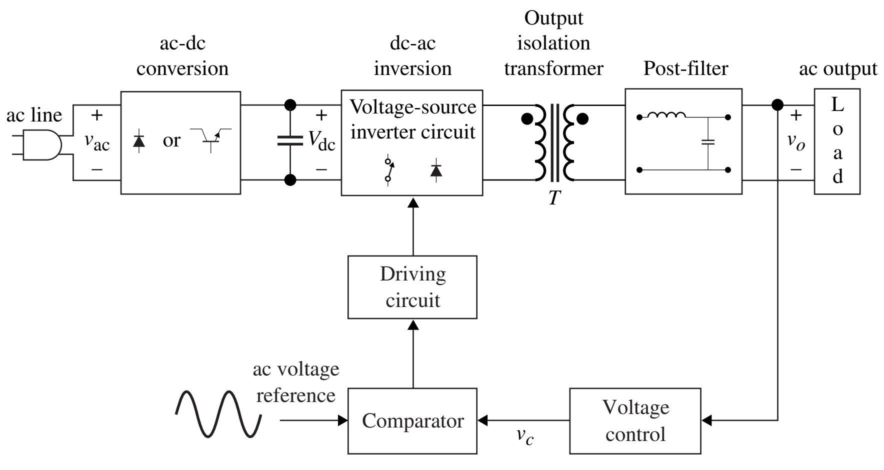


**Figure 9.1** Block diagram of a typical power electronic circuit with dc-to-ac inverter.

ac frequency is not necessarily the line frequency. Hence, if an SCR is used, we must devise a means to force it to turn off in order to produce the desired ac output. This feature, the variable-frequency output voltage, causes these circuits to play a very important role in all kinds of industrial applications, such as controlling the speed of motors. Variable-speed drives are used to control the speed of electric vehicles, pumps, rollers, and conveyors. Therefore, inverters circuits require more elaborate control signals to shape the ac voltage.

The load in Fig. 9.1 is broadly classified as either passive or active. If the load consists of impedance only (i.e., is passive), its time domain response is determined by the nature of the load and cannot be controlled externally. If the load consists of a source (i.e., is active), its time domain information can be controlled externally, as in the case of machine loads.

One final note should be made about power switches. The simplest inverter is one that employs a switching device that can be gate-controlled to interrupt current flow, resulting in a naturally commutating or self-commutated inverter. These switching devices are the gate-driven types such as GTOs, IGBTs, MOSFETs, and power BJTs. When SCRs are used, the converter requires an additional external circuit to force the SCR to turn off. The reason for the mandatory forced commutation in such inverters is that the dc input voltage across the SCR devices causes them to be in the forward conduction state. The higher the power rating of the gate-controlled devices, the less SCRs are used in the design of inverters.

## Voltage- and Current-Source Inverters

Since a practical source can provide either a constant voltage or a constant current, broadly speaking, inverters are divided into voltage-source inverters (VSIs) and current-source inverters (CSIs). The dc source in a VSI is a fixed voltage such as a battery, fuel cell, solar cell, dc generator, or rectified dc source. In a CSI, the dc source is a nearly constant current source. The nature of the source, whether it is a dc current source or a dc voltage source, makes the power inverter clearly distinguishable and its practical application more defined. Block diagrams for a voltage-source and a current-source inverter are shown in Fig. 9.2($a$) and ($b$), respectively.

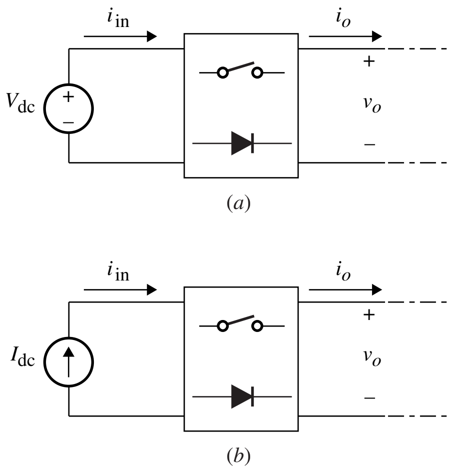


**Figure 9.2** Block diagram representations for ($a$) voltage source inverter and ($b$) current-source inverter.

In the voltage-source inverter (VSI), the output voltage, $v_o$, is a function of the inverter operation; the load current, $i_o$, is a function of the nature of the load; and the dc input, $V_{dc}$, is a constant input voltage. In the current-source inverter, the output voltage is a function of the inverter operation; the load current, $i_o$, is a function of the nature of the load; and the source, $I_{dc}$ is a constant input current.

## Inverter Configurations

Figure 9.3(*a*), (*b*), and (*c*) shows, respectively, three possible single-phase inverter arrangements: biphase, half-bridge, and full-bridge. The biphase inverter, also known as a push-pull inverter, is drawn in two different ways in Fig. 9.3(*a*).

## Output Voltage Control

If the output voltage is controlled by varying the dc source voltage, this can be accomplished by either controlling the dc input by using a front dc-dc converter, as shown in Fig. 9.4(*a*), or by using an ac-dc phase control converter, as shown in Fig. 9.4(*b*). In many applications, varying the input dc voltage is not possible or is costly.

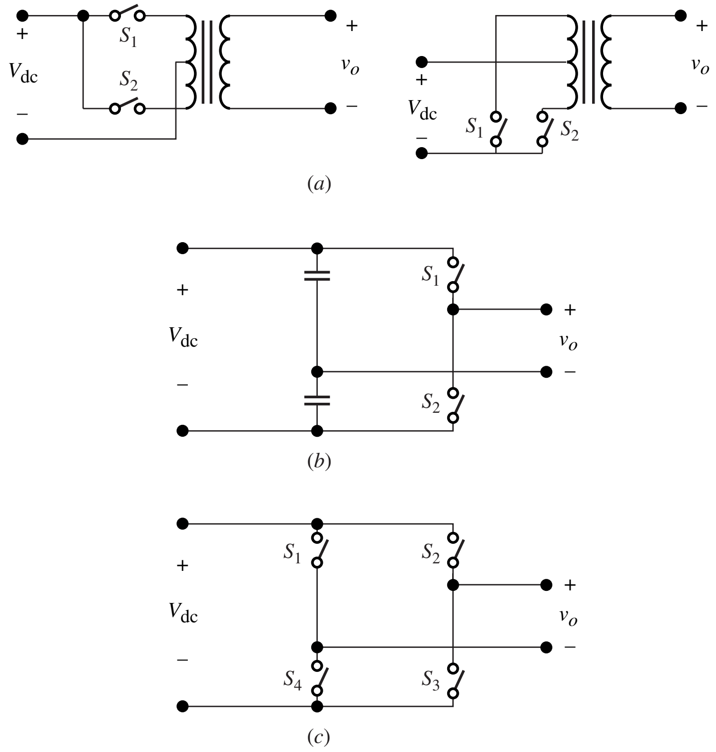


**Figure 9.3** Single-phase inverter arrangements. (*a*) Biphase inverter. (*b*) Half-bridge inverter. (*c*) Full-bridge inverter.

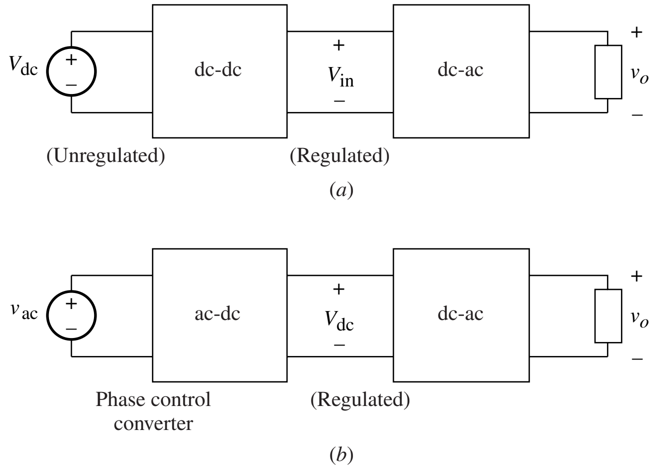


**Figure 9.4** Controlling the dc input using (*a*) dc-dc converter or (*b*) ac-dc phase-controlled rectifier.

## 9.2 BASIC HALF-BRIDGE INVERTER CIRCUITS

### 9.2.1 Resistive Load

To illustrate the basic concept of a dc-to-ac inverter circuit, we consider a half-bridge voltage-source inverter circuit under a resistive load as shown in Fig. 9.5(*a*). The switching waveforms for $S_1$, $S_2$, and the resultant output voltage are shown in Fig. 9.5(*b*).

The circuit operation is very simple. $S_1$ and $S_2$ are switched on and off alternatively at a 50% duty cycle, as shown in the switching waveform in Fig. 9.5(*b*). This shows that the circuit generates a square ac voltage waveform across the load from a

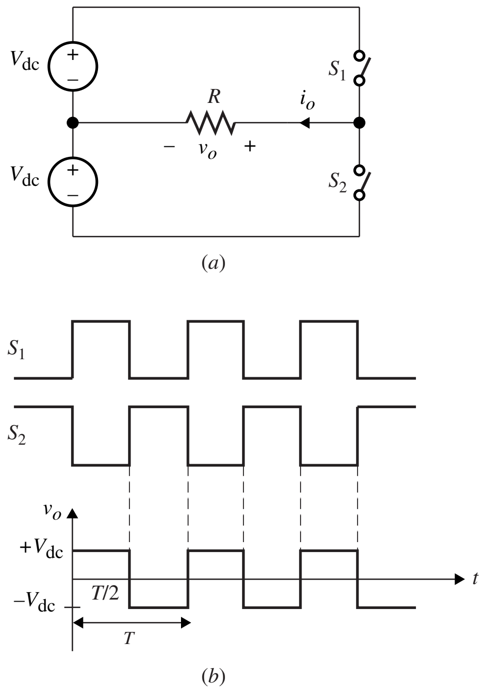


**Figure 9.5** (*a*) Half-bridge inverter under resistive load. (*b*) Switching and output voltage waveforms.

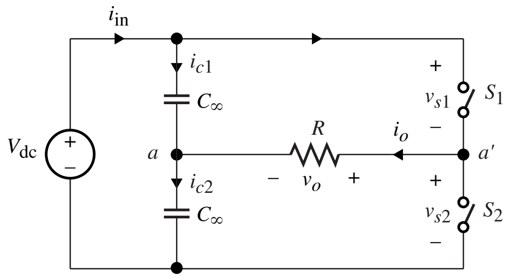


**Figure 9.6** Half-bridge inverter circuit with large splitting capacitors.

constant dc source. The voltages $V_{dc}$ and $-V_{dc}$ are applied across $R$ when $S_1$ is on and $S_2$ is off, and when $S_2$ is on and $S_1$ is off, respectively. One observation to be made here is that the frequency of the output voltage is equal to $1/T$ and is determined by the switching frequency. This is true as long as $S_1$ and $S_2$ are switched complementarily. Moreover, the rms value of the output voltage is simply $V_{dc}$. Hence, to control the rms value of the output voltage, we must control the rectified $V_{dc}$ voltage source. Another observation is that the load power factor is unity since we have a purely resistive load. This is rarely encountered in practical applications.

Finally, we should note that in practice the circuit does not require two equal dc voltage sources as shown in Fig. 9.5(*a*). Instead, large splitting capacitors are used to produce two equal dc voltage sources as shown in Fig. 9.6. The two capacitors, $C_\infty$, are equal and very large, so that $RC_\infty$ is much larger than the half switching period. This will guarantee that the midpoint, $a$, between the capacitors has a fixed potential at one-half of the supply voltage $V_{dc}$. The current from the source, $V_{dc}$, equals one-half of the load current, $i_o$. In steady state, the average capacitor currents are zero; hence, capacitors $C_\infty$ are used to block the dc component of $i_o$.

The limitation of this half-bridge configuration is that varying the switching sequence cannot control the output voltage.

## EXAMPLE 9.1

Sketch the current and voltage waveforms for $i_{in}$, $i_o$, $v_{s1}$, and $v_{s2}$, for the circuit shown in Fig. 9.6 for $\theta = 0$ and $\theta \neq 0$ by using the switching waveforms for $S_1$ and $S_2$ shown in Fig. 9.7(*a*). Determine the average output voltage in terms of $V_{dc}$ and $\theta$ when the inverter operates in the steady state.

***SOLUTION*** Let us consider mode 1, when $S_1$ is on and $S_2$ is off. Then the output voltage and current equations are given by

$$v_o = \frac{V_{dc}}{2}$$

$$i_o = \frac{v_o}{R} = \frac{V_{dc}}{2R}$$

Since we assumed that the capacitors are equal, the load current, $i_o$, splits equally, i.e.,

$$i_{c1} = -\frac{1}{2}i_o = -\frac{V_{dc}}{4R}$$

$$i_{c2} = i_{in} = -i_{c1} = \frac{V_{dc}}{4R}$$

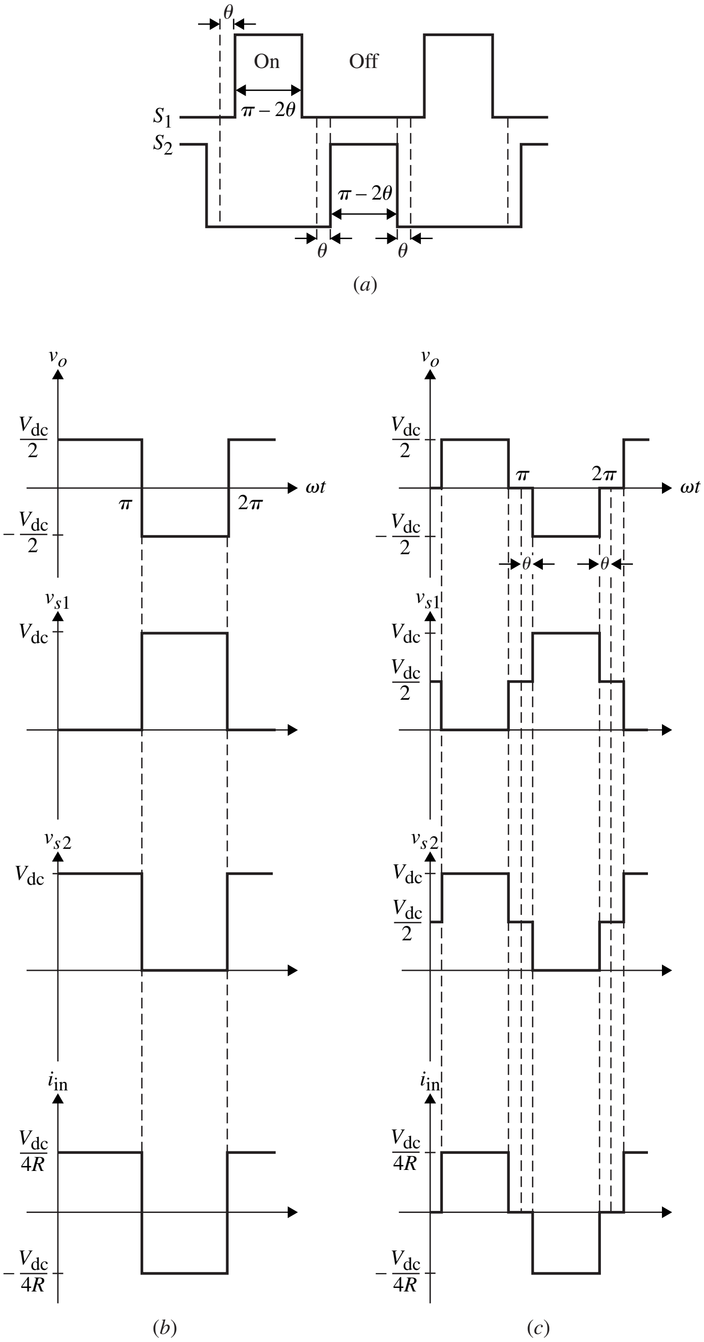


**Figure 9.7** (*a*) Switching waveforms for Example 9.1. (*b*) Current and voltage waveforms for $\theta = 0$. (*c*) Current and voltage waveforms for $\theta \neq 0$.

The voltages across the switches are

$$v_{s1} = 0$$

$$v_{s2} = v_o + \frac{V_{dc}}{2} = V_{dc}$$

Mode 2 starts when $S_1$ and $S_2$ are off during the short interval $\theta$.

$$v_o = 0$$

$$i_o = 0$$

$$i_{c1} = i_{c2} = i_{\text{in}} = 0$$

$$v_{s1} = v_{s2} = \frac{V_{\text{dc}}}{2}$$

Mode 3 starts when $S_2$ is on and $S_1$ is off, which yields the following equations:

$$v_o = -\frac{V_{\text{dc}}}{2}$$

$$i_o = \frac{v_o}{R} = -\frac{V_{\text{dc}}}{2R}$$

$$i_{c1} = \frac{i_o}{2} = -\frac{V_{\text{dc}}}{4R}$$

$$i_{c2} = -i_{c1} = \frac{V_{\text{dc}}}{4R}$$

$$i_{\text{in}} = i_{c1} = -\frac{V_{\text{dc}}}{4R}$$

$$v_{s1} = \frac{V_{dc}}{2} - v_o = V_{\text{dc}}$$

$$v_{s2} = 0$$

Mode 4 is similar to mode 2 since both switches are open.
The average output voltage is given by

$$v_{o,\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v_o^2(t)\ dt}$$

$$v_{o,\text{rms}} = \sqrt{\frac{1}{2\pi}\left[\int_\theta^{\pi-\theta}\left(\frac{V_{\text{dc}}}{2}\right)^2 d\theta + \int_{\pi+\theta}^{2\pi-\theta}\left(\frac{-V_{\text{dc}}}{2}\right)^2 d\theta\right]} \tag{9.1}$$

$$v_{o,\text{rms}} = \frac{V_{\text{dc}}}{2}\sqrt{\left(1 - \frac{2\theta}{\pi}\right)}$$

Notice that when $\theta = 0$, the rms value of the output is $V_{\text{dc}}/2$, as expected. Notice also that in a practical situation under an inductive load, the two switches are not allowed to switch off simultaneously. Figure 9.7(*b*) and (*c*) shows the currents and voltages for $\theta = 0$ and $\theta \neq 0$, respectively. Since the waveform is symmetric, the average output voltage will always be zero.

### 9.2.2 Inductive-Resistive Load

Figure 9.8(*a*) shows a half-bridge inverter under an inductive-resistive load, with its equivalent circuit and the output waveforms shown in Fig. 9.8(*b*) and (*c*), respectively.

With $S_1$ and $S_2$ switched complementarily, each at a 50% duty cycle at a switching frequency *f*, then the load between terminals *a* and $a'$ is excited by a square voltage

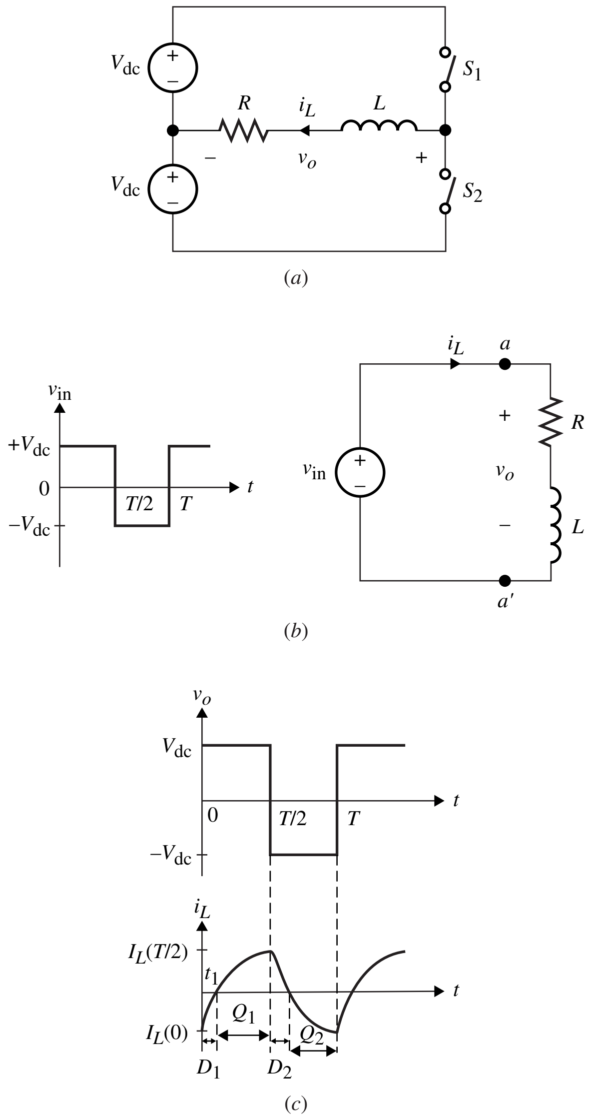


**Figure 9.8** (*a*) Half-bridge inverter with inductive-resistive load. (*b*) Equivalent circuit. (*c*) Steady-state waveforms.

waveform $v_{in}(t)$ of amplitudes $+V_{dc}$ and $-V_{dc}$ as shown in Fig. 9.8(*b*); i.e., $v_{in}(t)$ is defined as follows:

$$v_{in}(t) = \begin{cases} +V_{dc} & 0 \le t < T/2 \\ -V_{dc} & T/2 \le t < T \end{cases} \tag{9.2}$$

The switches are implemented using a conventional SCR (requiring an external forced commutation circuit) or fully controlled power switching devices such as IGBTs, GTOs, BTJs, or MOSFETs. Notice that from the direction of the load current $i_L$, these switches must be bidirectional. An example is the half-bridge inverter circuit shown in Fig. 9.9 with $S_1$ and $S_2$ implemented by MOSFETs.

Assume the inverter operates in the steady state and its inductor current waveform is shown in Fig. 9.8(*c*). For $0 \le t < t_1$, the inductor current is negative, which means that while $S_1$ is on the current actually flows in the reverse direction, i.e., in the

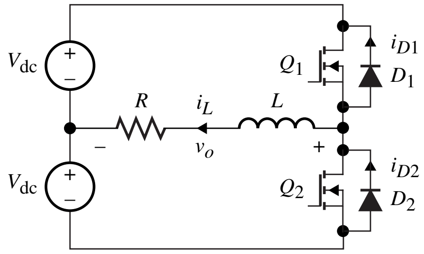


**Figure 9.9** MOSFET implementation for $S_1$ and $S_2$ in the half-bridge inverter of Fig. 9.8(*a*).

body (flyback) diode of the bidirectional switch $S_1$. At $t = t_1$, the current flows through transistor $Q_1$, as shown in Fig. 9.8(*c*). At $t = T/2$, when $S_2$ is turned on, since the current direction is positive, the flyback diode, $D_2$, turns on until $t = T/2 + t_1$, when $Q_2$ starts conducting.

In steady state, the following conditions must hold:

$$i_L(0) = -i_L(T/2)$$
$$i_L(0) = i_L(T)$$

During the first interval $(0 \le t < T/2)$, when $S_1$ is on and $S_2$ is off, $v_{in}(t) = +V_{dc}$, resulting in the following equation for $i_L(t)$:

$$L\frac{di_L}{dt} + Ri_L = V_{dc} \tag{9.3}$$

If the inductor initial value equals $I_L(0)$, the solution for $i_L(t)$ is given by

$$i_L(t) = -\left(I_L(0) + \frac{V_{dc}}{R}\right)e^{-t/\tau} + \frac{V_{dc}}{R} \tag{9.4}$$

where $\tau = L/R$. Since $I_L(T/2) = -I_L(0)$, the initial condition at $t = 0$ is constant and given by

$$I_L(0) = -\frac{V_{dc}}{R}\frac{1 - e^{-T/2\tau}}{1 + e^{-T/2\tau}} \tag{9.5}$$

The second half-cycle for $t > T/2$ produces the following expression for $i_L(t)$ with the initial condition at $t = T/2$ equaling $-I_L(0)$:

$$i_L(t) = \left(I_L(0) + \frac{V_{dc}}{R}\right)e^{-(t - T/2)/\tau} - \frac{V_{dc}}{R} \tag{9.6}$$

This expression equals $-i_L(t)$ of Eq. (9.4), which is valid for the interval $(T/2 \le t < T)$.

The exact expression for the average power delivered to the load can be obtained from the following relation:

$$\begin{aligned} P_{o,\text{ave}} &= \frac{1}{T}\int_0^T i_L(t)v_o(t)\,dt \\ &= \frac{2V_{dc}}{T}\int_0^{T/2}\left[-\left(I_L(0) + \frac{V_{dc}}{R}\right)e^{-t/\tau} + \frac{V_{dc}}{R}\right]dt \end{aligned} \tag{9.7}$$

where $I_L(0)$ is given by Eq. (9.5).

To determine the device's ratings, we must find the average and rms current values through the switching devices and flyback diodes. It is clear that when $i_L$ changes polarity at $t = t_1$ during the first interval, the load current changes polarity and, hence, commutates from $D_1$ to $Q_1$. Similarly, at $t = t_1 + T/2$, the current commutation goes from $D_2$ to $Q_2$ as shown in Fig. 9.8($c$). The time at which $i_L(t)$ becomes zero, $t = t_1$, is obtained by setting $i_L(t)$ in Eq. (9.4) to zero at $t = t_1$, to yield

$$t_1 = \tau \ln \frac{2}{1 + e^{-T/2\tau}} \tag{9.8}$$

Obtaining the value of $t_1$ allows us to find the average and rms values for the switching devices and flyback diodes.

### Average Transistor and Diode Currents

To help us obtain quantitatively the expressions for the diode and transistor currents, we represent the load voltage and current by their fundamental components as shown in Fig. 9.10($a$).

Let the fundamental components of $v_o(t)$ and $i_L(t)$ be given by

$$v_{o1}(t) = V_{o1} \sin \omega t \tag{9.9}$$

$$i_{L1}(t) = I_{o1} \sin(\omega t + \theta) \tag{9.10}$$

where $V_{o1} = 2V_{\text{dc}}/\pi$, and $I_{o1}$ and $\theta$ are the peak and the phase angle of $i_{L1}(t)$ as shown in Fig. 9.10($a$), which are given by

$$I_{o1} = \frac{2V_{\text{dc}}}{\pi|Z|}, \quad \theta = \tan^{-1}\left(\frac{\omega L}{R}\right) \tag{9.11}$$

where $|Z| = \sqrt{R^2 + (\omega L)^2}$. Using the fundamental component expression, the rms values for each of the diode and transistor currents are given by

$$\begin{aligned} I_{D,\text{rms}} &= \sqrt{\frac{1}{2\pi}\int_0^{\theta} i_{L1}^2(t)\, d\omega t} \\ &= \sqrt{\frac{1}{2\pi}\int_0^{\theta} I_{o1}^2 \sin^2 \omega t\, d\omega t} \\ &= \sqrt{\frac{I_{o1}^2}{4\pi}\left(-\frac{\sin 2\theta}{2} + \theta\right)} = \frac{I_{o1}}{2}\sqrt{\frac{\theta}{\pi} - \frac{\sin\theta\cos\theta}{2\pi}} \end{aligned} \tag{9.12}$$

$$\begin{aligned} I_{Q,\text{rms}} &= \sqrt{\frac{1}{2\pi}\int_0^{\pi} I_{o1}^2 \sin^2 \omega t\, d\omega t} \\ &= \sqrt{\frac{I_{o1}^2}{4\pi}(\pi + \cos\theta\sin\theta - \theta)} = \frac{I_{o1}}{2}\sqrt{1 - \frac{\theta}{\pi} + \frac{\sin\theta\cos\theta}{\pi}} \end{aligned} \tag{9.13}$$

and the average values for each of the diode and transistor currents are given by

$$\begin{aligned} I_{D,\text{ave}} &= \frac{1}{2\pi}\int_0^{\theta} I_{o1} \sin \omega t\, d\omega t \\ &= \frac{I_{o1}}{2\pi}(1 - \cos\theta) \end{aligned} \tag{9.14}$$

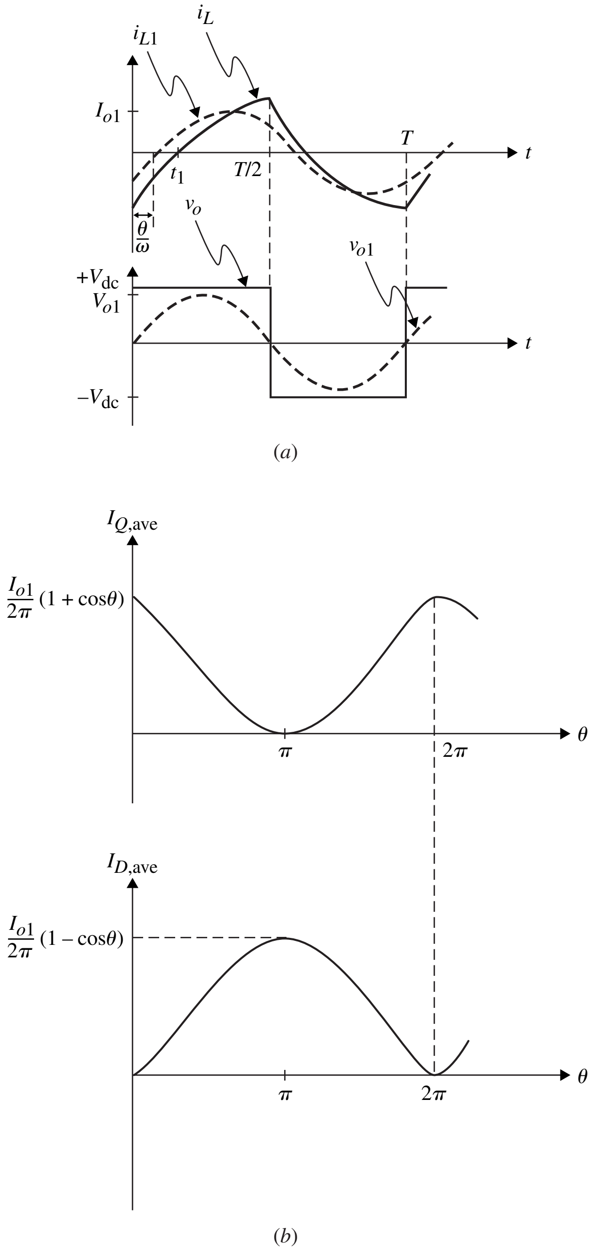


**Figure 9.10** (*a*) Output voltage and current waveforms. (*b*) Average transistor and diode current waveforms as a function of $\theta$.

$$I_{Q,\text{ave}} = \frac{I_{o1}}{2\pi}(1 + \cos\theta) \tag{9.15}$$

Notice that under a resistive load, the lagging phase angle is zero and the entire load current flows through $Q_1$ and $Q_2$ with $D_1$ and $D_2$ always reverse biased. This is clear from the zero average diode current obtained from the preceding equation. Under a purely inductive load, the phase angle is 90° and the load current flows through the diode and transistors for an equal time, resulting in equal average and rms current values for the diode and transistor. Figure 9.10(*b*) shows the average transistor and diode current values as functions of the phase angle.

The average output power delivered to the load is approximated by

$$P_{o,\text{ave}} = V_{o1,\text{rms}} I_{o1,\text{rms}} \cos\theta \tag{9.16}$$

where

$$V_{o1,\text{rms}} = \frac{V_{o1}}{\sqrt{2}} = \frac{\sqrt{2}V_{\text{dc}}}{\pi}$$

$$I_{o1,\text{rms}} = \frac{I_{o1}}{\sqrt{2}}$$

Equation (9.16) yields the following expression for $P_{o,\text{ave}}$:

$$P_{o,\text{ave}} = \frac{2V_{\text{dc}}^2}{\pi^2|Z|}\cos\theta \tag{9.17}$$

The ripple voltage in the traditional dc-to-ac converter is defined by

$$V_{o,\text{ripple}} = \sqrt{V_{o,\text{rms}}^2 - V_{o,\text{ave}}^2} \tag{9.18}$$

Similarly, the input ripple current expression in the dc-to-ac inverter is defined by

$$I_{\text{in,ripple}} = \sqrt{I_{o,\text{rms}}^2 - I_{o,\text{ave}}^2} \tag{9.19}$$

$$I_{\text{in,ripple}} \approx I_{o1,\text{rms}} = \frac{\sqrt{2}V_{\text{dc}}}{\pi|Z|} \tag{9.20}$$

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**EXAMPLE 9.2**

Consider the half-bridge inverter of Fig. 9.9 with the following circuit components: $V_{\text{dc}} = 408$ V, $R = 8\ \Omega$, $f = 400$ Hz, and $L = 40$ mH.

**(a)** Derive the exact expression for $i_L(t)$.

**(b)** Derive the expression for the fundamental component of $i_L(t)$.

**(c)** Determine the average diode and transistor currents.

**(d)** Determine the average power delivered to the load.

***SOLUTION*** **(a)** The exact solution for $i_L(t)$ was derived before, and is given again in Eq. (9.21).

$$i_L(t) = \begin{cases} -\left(I_L(0) + \dfrac{V_{\text{dc}}}{R}\right)e^{-t/\tau} + \dfrac{V_{\text{dc}}}{R} & 0 \le t \le T/2 \\ +\left(I_L(0) + \dfrac{V_{\text{dc}}}{R}\right)e^{-(t-T/2)/\tau} - \dfrac{V_{\text{dc}}}{R} & T/2 \le t \le T \end{cases} \tag{9.21}$$

Since $i_L(T/2) = -I_L(0)$, we have

$$I_L(0) = -\frac{V_{\text{dc}}}{R}\frac{1-e^{-T/2\tau}}{1+e^{-T/2\tau}} \quad \text{where } \tau = \frac{L}{R} \tag{9.22}$$

**(b)** To calculate the fundamental component of $i_L(t)$, we first determine the $a_1$ and $b_1$ coefficients:

$$I'_{L1}(t) = \frac{2}{T}\left[\int_0^{T/2} i_L(t)\cos\omega t\ dt + \int_{T/2}^{T} i_L(t)\cos\omega t\ dt\right] \tag{9.23a}$$

$$I''_{L1}(t) = \frac{2}{T}\left[\int_0^{T/2} i_L(t)\sin\omega t\ dt + \int_{T/2}^{T} i_L(t)\sin\omega t\ dt\right] \tag{9.23b}$$

The fundamental component of $i_L(t)$ is given by

$$i_{L1}(t) = I'_{L1}(t)\cos\omega t + I''_{L1}(t)\sin\omega t \tag{9.24}$$

$$i_{L1}(t) = -0.57\cos(2513.3t + 57.8°)\ \text{A}$$

The rms value of $i_L(t)$ is

$$I_{L,\text{rms}} = \sqrt{\frac{1}{T}\left[\int_0^{T/2} i_L^2(t)\ dt + \int_{T/2}^{T} i_L^2(t)\ dt\right]}$$

$$I_{L,\text{rms}} = 64.1\ \text{A}$$

**(c)** The average diode current is given by Eq. (9.14) as

$$I_{D,\text{ave}} = \frac{I_{o1}}{2\pi}(1 - \cos\theta)$$

with $\theta = \tan^{-1}(\omega L/R) = 85.45°$ and

$$I_{o1} = \frac{2V_{\text{dc}}}{\pi|Z|} = \frac{2\times 408}{\pi\times 100.85} = 2.58\ \text{A}$$

$$|Z| = \sqrt{R^2 + (\omega L)^2} = 100.85\ \Omega$$

If we substitute these values, then $I_{D,\text{ ave}}$ turns out to be

$$I_{D,\text{ave}} = \frac{2.58}{2\pi}(1 - \cos 85.45°) = 0.38\ \text{A}$$

Similarly, $I_{Q,\text{ave}}$ is equal to

$$I_{Q,\text{ave}} = \frac{I_{o1}}{2\pi}(1 + \cos\theta) = 0.44\ \text{A}$$

**(d)** The average power is given by Eq. (9.17):

$$P_{\text{ave}} = V_{o1,\text{rms}}I_{o1,\text{rms}}\cos\theta$$

$$= \frac{2V_{\text{dc}}^2}{\pi^2|Z|}\cos\theta$$

$$P_{\text{ave}} = \frac{2(408)^2}{\pi^2(100.85)}\cos(85.45) = 26.53\ \text{W}$$

---

## EXERCISE 9.1

Figure E9.1 shows a half-bridge inverter under a purely inductive load with $V_{\text{dc}} = 24$ V, $L = 10$ mH, and $f = 60$ Hz. Assume $Q_1$ and $Q_2$ are operating at a 50% duty cycle and the circuit has reached the steady-state operation.

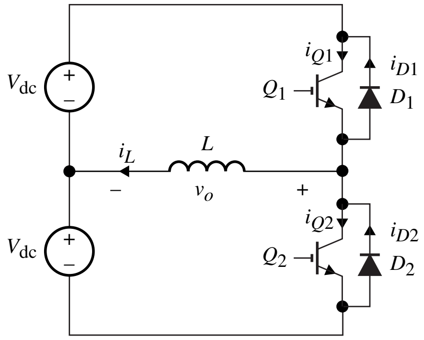



**Figure E9.1** Half-bridge inverter with purely inductive load.

**(a)** Sketch the waveforms for $i_L$, $i_{Q1}$, $i_{D1}$, $i_{Q2}$, $i_{D2}$, and $v_o$.
**(b)** Calculate the rms value of $i_L(t)$.
**(c)** Determine the average output power delivered to the load.

***ANSWER*** $I_{L,\text{rms}} = 11.54\text{ A}$, $P_{\text{ave}} = 0\text{ W}$

## EXERCISE 9.2

Consider the single-phase inverter of Fig. 9.8(*a*) with an inductive-resistive load that delivers 400 W to a 60 Hz load from a dc source of 420 V. Assume the output voltage and current are represented by their fundamental components with a lagging power factor of 0.6. Determine the average and rms current values for the transistors and diodes.

***ANSWER*** $I_{D,\text{rms}} = 0.81$ A, $I_{D,\text{ave}} = 0.45$ A, $I_{Q,\text{rms}} = 2.31$ A, $I_{Q,\text{ave}} = 1.80$ A

## EXAMPLE 9.3

Draw the output voltage and $v_{s1}$ waveforms for the center-tap biphase inverter shown in Fig. 9.11. Assume $S_1$ and $S_2$ are bidirectional switches and are switched at a 50% duty cycle. It is used in a low-input-voltage application to reduce losses, since the current only flows half-period in a section of the transformer (the transformer is not fully utilized). The two modes of operations are shown in Fig. 9.11(*b*) and (*c*); the waveforms are shown in Fig. 9.11(*d*).

***SOLUTION*** The equivalent circuit for mode 1, when switch $S_1$ is on, is shown in Fig. 9.11(*b*). The output voltage is obtained as follows:

$$\frac{v_o}{V_{dc}} = \frac{n_2}{n_1}$$

$$v_o = \frac{n_2}{n_1} V_{dc}$$

Figure 9.11(*c*) shows the equivalent circuit for mode 2, with $v_o$ given by

$$\frac{v_o}{V_{dc}} = -\frac{n_2}{n_1}$$

$$v_o = -\frac{n_2}{n_1} V_{dc}$$

The waveforms for $v_o$ are shown in Fig. 9.11(*d*).

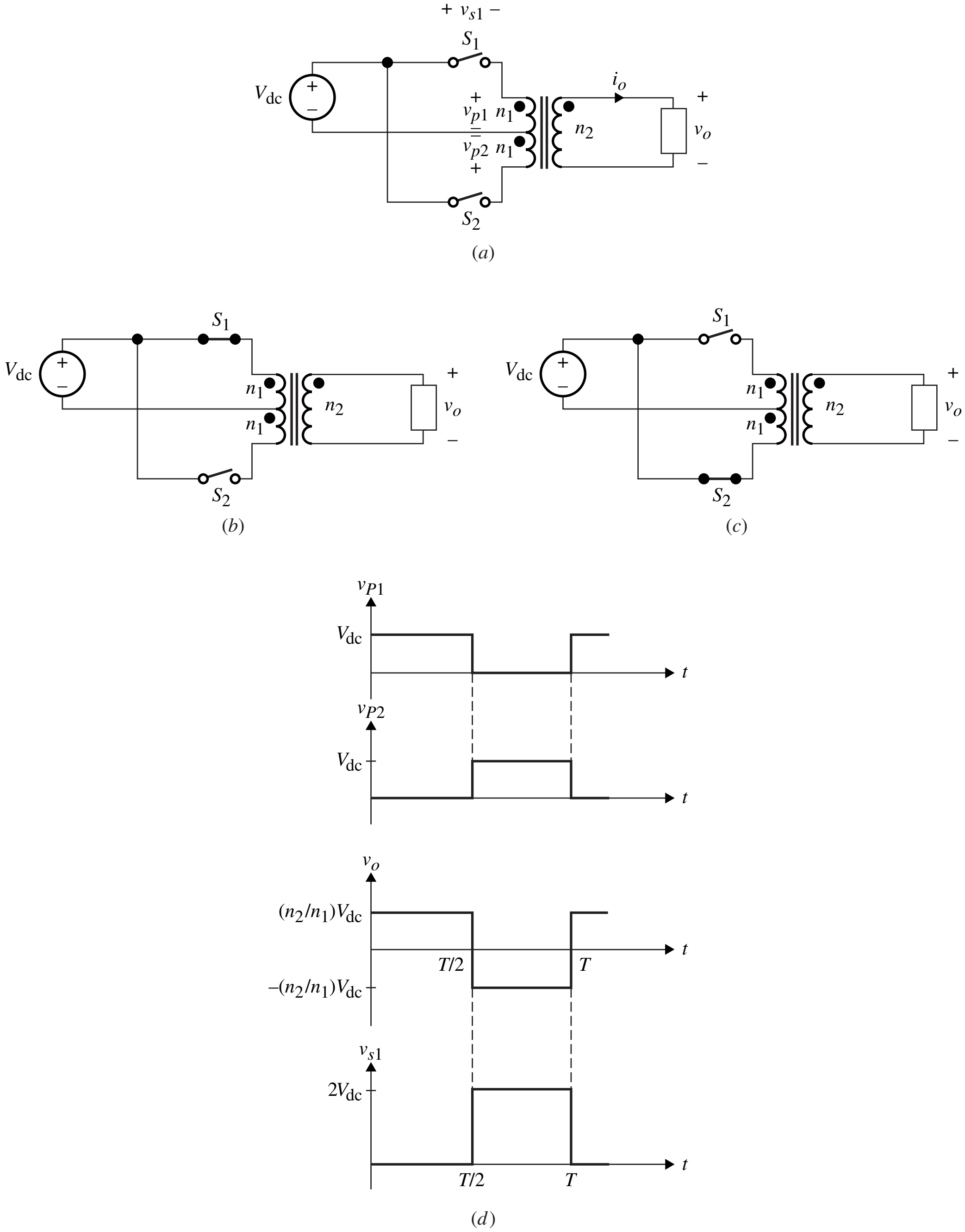



**Figure 9.11** (*a*) Center-tap biphase inverter for Example 9.3. (*b*) Mode 1. (*c*) Mode 2. (*d*) Voltage waveforms.

## 9.3 FULL-BRIDGE INVERTERS

Figure 9.12 shows the full-bridge circuit configuration for a voltage-source inverter under resistive load.

Depending on the switching sequence of $S_1$, $S_2$, $S_3$, and $S_4$, the output voltage can be controlled either by varying $V_{dc}$ only or by controlling the phase shift between the

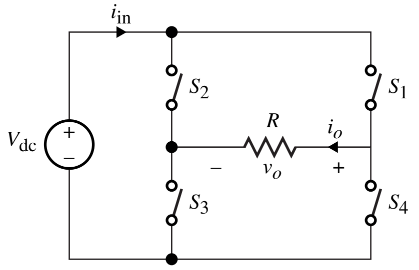



**Figure 9.12** Full-bridge inverter under a purely resistive load.

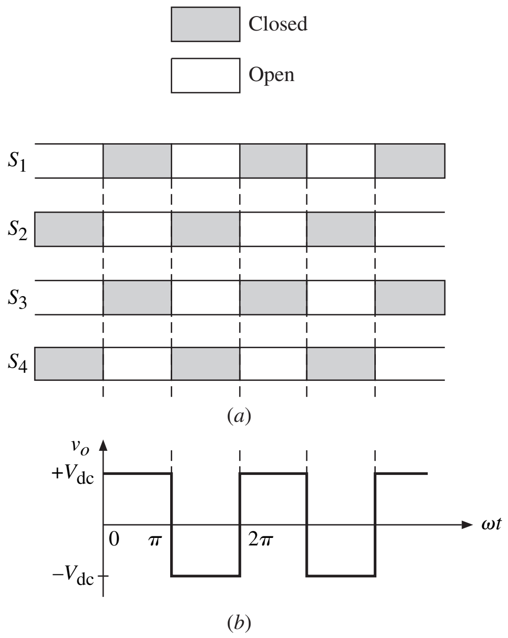



**Figure 9.13** (*a*) Switching sequence for full-bridge voltage-source inverter at 50% duty cycle. (*b*) Output voltage waveform.

switches. If $S_1$-$S_3$ and $S_2$-$S_4$ are switched on and off at a 50% duty cycle as shown in Fig. 9.13(*a*), the output voltage, shown in Fig. 9.13(*b*), is a symmetrical squarewave whose fundamental rms value is controlled by varying only $V_{dc}$. The fundamental value of $v_o(t)$ is given by

$$v_{o1}(t) = \frac{V_{dc}}{\pi} \sin \omega t \tag{9.25}$$

The rms value of the fundamental component is $V_{dc}/\sqrt{2}\pi$. The use of SCR rectifier circuits or dc-dc switch-mode converters normally accomplishes varying of the dc voltage source.

Another method to control the rms value of the output voltage is to use the switching sequence shown in Fig. 9.14(*a*). This switching sequence is obtained by shifting the timing sequence of $S_1$ and $S_4$ in Fig. 9.13(*a*) to the left by a phase angle $\alpha$, and $S_2$ and $S_3$ to the right by the same angle.

The fundamental component of the output voltage, $v_{o1}(t)$ is given by

$$v_{o1}(t) = V_{dc}\sqrt{\frac{1}{2} - \frac{\alpha}{\pi}} \sin(\omega t - \theta) \tag{9.26}$$

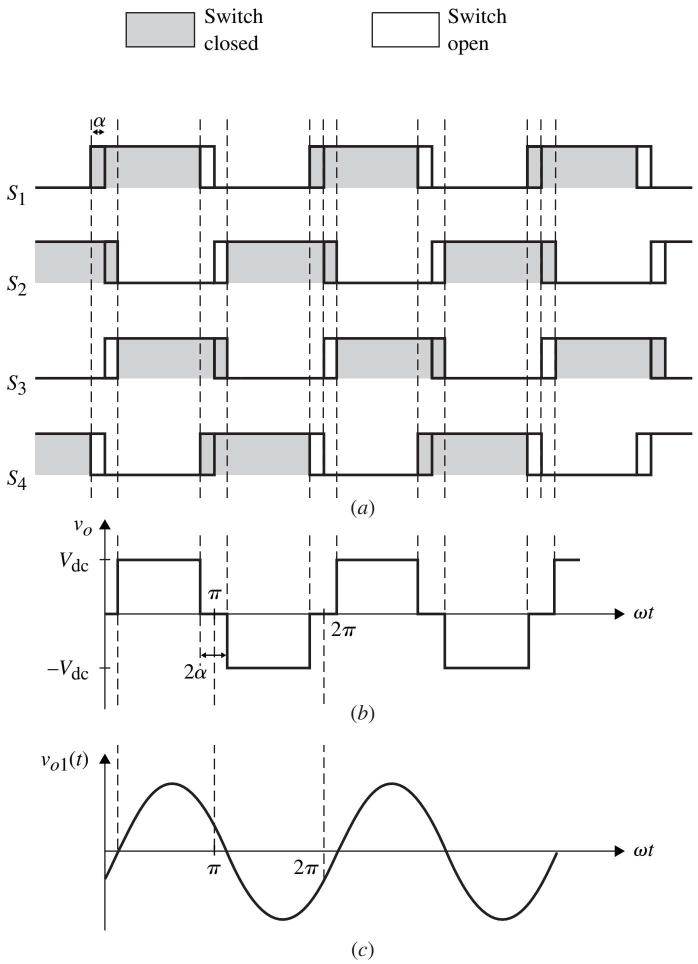



**Figure 9.14** (*a*) Switching sequence with $\alpha$ phase shift. (*b*) Output voltage. (*c*) Fundamental component for $v_o(t)$.

and the rms value is given by $V_{dc}\sqrt{\frac{1}{4} - \alpha/(2\pi)}$. It is clear from this relation that the rms value or the peak of the fundamental component can be controlled by the amount of the phase shift between the switching signals. We notice that, unlike the switching sequence given in Fig. 9.13(*b*), which has two states for the output voltage—$+V_{dc}$ and $-V_{dc}$—the switching sequence in Fig. 9.14(*a*) gives three states for the output voltage, $+V_{dc}$, 0, and $-V_{dc}$; such inverters are known as *tri-state inverters*. Since the load is resistive, the four switches can be implemented by SCRs with a unidirectional current flow. This is because the load current can reverse direction instantaneously as the voltage $v_o$ reverses its direction. However, under an inductive load, for the circuit to work using SCRs, a diode must be added in parallel with each SCR as shown in Fig. 9.15.

It can be seen from the switching sequence of Fig. 9.14(*a*) that in the steady state, there are four modes of operation, as shown in Fig. 9.16. The switch implementation for $S_1$, $S_2$, $S_3$, and $S_4$ is quite simple since there is no need for bidirectional current flow. The average power delivered to the load is $(V_{o,\text{rms}}^2/R)$, where $V_{o,\text{rms}}$ is equal to $V_{dc}$ or $V_{dc}\sqrt{\frac{1}{4} - \alpha/2\pi}$, depending on whether the switching sequence of Fig. 9.13(*a*) or Fig. 9.14(*a*), respectively, is used.

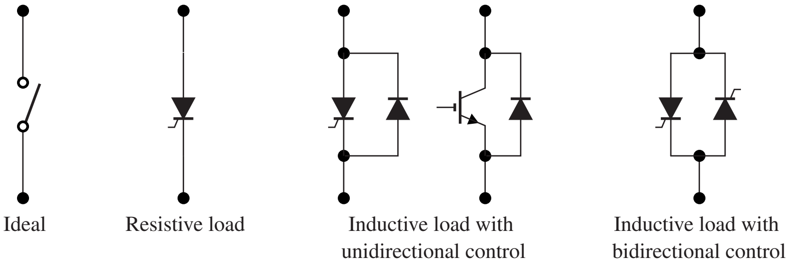



**Figure 9.15** Possible switch implementation.

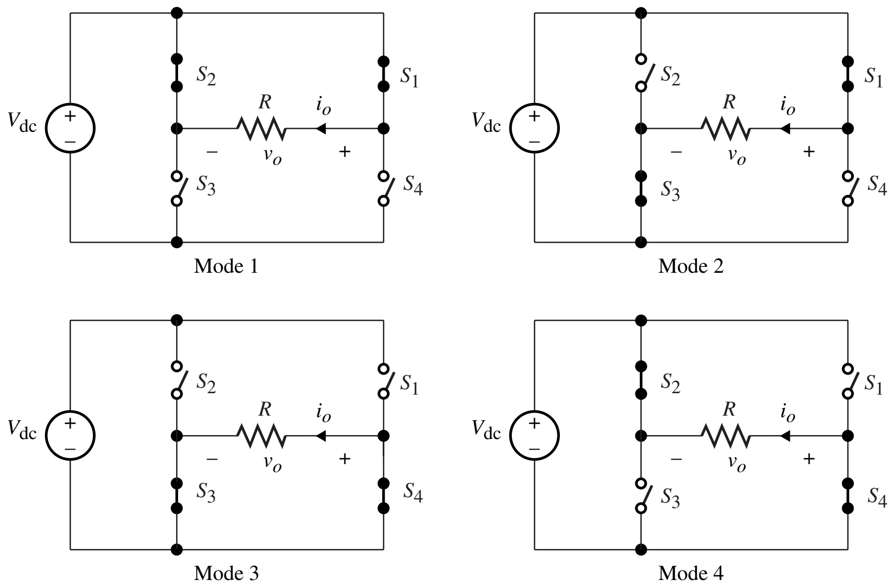



**Figure 9.16** Modes of operation.

## EXAMPLE 9.4

Consider the resistive-load full-bridge voltage-source inverter shown in Fig. 9.12 with the following circuit parameters: $V_{dc} = 150$ V, $R = 12\ \Omega$, and $f_s = 60$ Hz. Sketch the waveforms for $v_o$ and $i_{in}$ and determine the average power delivered to the load for the two switching sequences shown in Fig. 9.13(*a*) and Fig. 9.14(*a*), with $\alpha = 10°$.

***SOLUTION*** For the switch sequence shown in Fig. 9.13(*a*), $v_o$ and $i_o$ are symmetric and given by

$$v_o = \begin{cases} +V_{dc} & 0 \le t < T/2 \\ -V_{dc} & T/2 \le t < T \end{cases}$$

$$i_o = \begin{cases} +\dfrac{V_{dc}}{R} & 0 \le t < T/2 \\ -\dfrac{V_{dc}}{R} & T/2 \le t < T \end{cases}$$

The average output power is given by

$$P_{o,\text{ave}} = \frac{1}{T}\int_0^T v_o i_o \, dt$$

For the switch sequence shown in Fig. 9.14(*a*), the average output power is given by

$$P_{o,\text{ave}} = \frac{V_{o,\text{rms}}^2}{R}$$

where the rms value is expressed as

$$V_{o,\text{rms}} = V_{\text{dc}}\sqrt{\frac{1}{4} - \frac{\alpha}{2\pi}}$$

The resultant average output power for $\alpha = 10°$ is given by

$$P_{o,\text{ave}} = \frac{V_{\text{dc}}^2\left(1 - \frac{2\alpha}{\pi}\right)}{R}$$

$$= 1666.67 \text{ W}$$

---

As stated earlier, practical loads do not consist of a simple resistor with a unity power factor, but rather have some sort of an inductance. Figure 9.17(*a*) shows a full-bridge inverter under an inductive-resistive load. If the switches are operating at a 50% duty cycle with a two-state output, then the current and voltage waveforms are as shown in Fig. 9.17(*b*). The analysis of this inverter is similar to that for the half-bridge voltage-source inverter discussed earlier.

To obtain an expression for the average power absorbed by the load, we revert to using the Fourier series as our analysis technique. The rms values for $v_o$ and $i_o$ of Fig. 9.17(*b*) are based on a 50% squarewave output given by

$$I_{o,\text{rms}}^2 = I_{1,\text{rms}}^2 + I_{2,\text{rms}}^2 + \cdots + I_{n,\text{rms}}^2 \qquad (9.27a)$$

$$V_{o,\text{rms}}^2 = V_{\text{dc}} \qquad (9.27b)$$

where $I_{n,\text{rms}} = I_n/\sqrt{2}$ and $I_n$ is the peak current of the $n$th harmonic of $i_o(t)$.

Since the output voltage is a squarewave with a 50% duty cycle, its Fourier series may be expressed as follows:

$$v_o(t) = \frac{4V_{\text{dc}}}{\pi}\left(\sin\omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5} + \cdots + \frac{\sin n\omega t}{n}\right) \qquad (9.28)$$

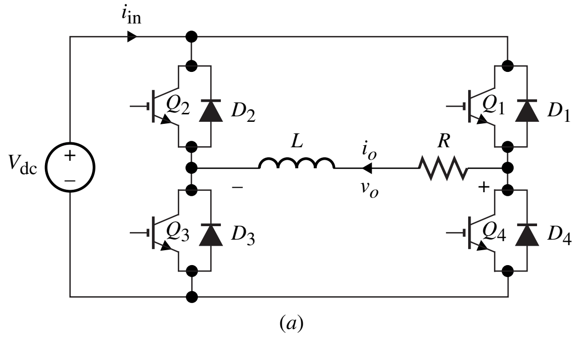
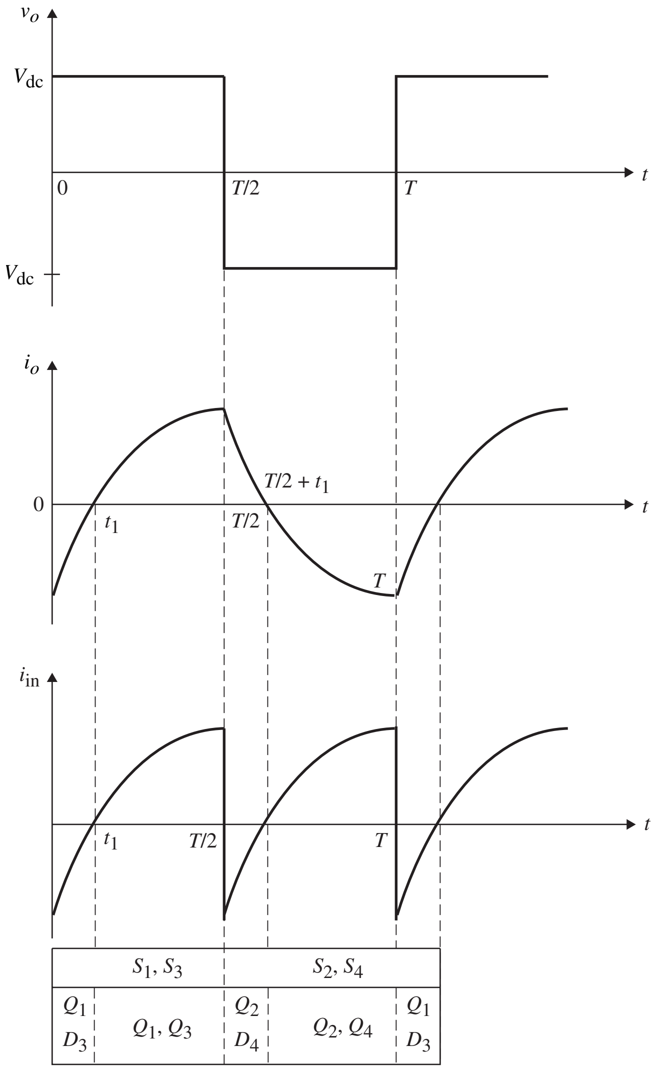
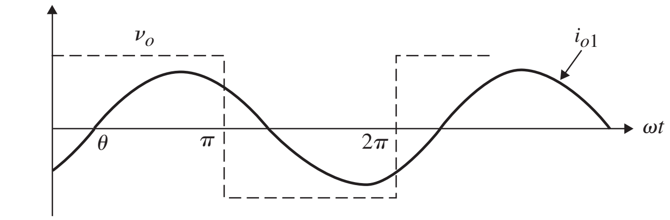



**Figure 9.17** (*a*) Full-bridge inverter under *R*-*L* load.




(*b*)




(*c*)

**Figure 9.17** (*continued*) (*b*) Waveforms under 50% duty cycle. (*c*) Fundamental component of the inductor current.

Therefore, $i_o(t)$ is given by

$$i_o(t) = \frac{4V_{dc}}{\pi}\left[\frac{\sin\omega t}{\sqrt{R^2+(\omega L)^2}} + \frac{\sin 3\omega t}{3\sqrt{R^2+(3\omega L)^2}} + \frac{\sin 5\omega t}{5\sqrt{R^2+(5\omega L)^2}} + \cdots + \frac{\sin n\omega t}{n\sqrt{R^2+(n\omega L)^2}}\right] \tag{9.29}$$

and the rms value for the $n$th current component is given by

$$I_{n,\text{rms}} = \frac{2\sqrt{2}V_{dc}}{n\pi|Z_n|}$$

where

$$|Z_n| = \sqrt{R^2+(n\omega L)^2}$$

If we assume that the major part of the average output power is delivered at the fundamental frequency, then $P_{o,\text{ ave}}$ is given by

$$P_{o,\text{ave}} = \frac{8V_{dc}^2}{\pi^2\sqrt{R^2+(\omega L)^2}} \tag{9.30}$$

### 9.3.1 Approximate Analysis

An approximate solution for the load current can be obtained by assuming that $L/R >> T/2$. This will allow us to represent the load current by its first harmonic. As illustrated in the previous example, Fig. 9.17($c$) shows the inductor current represented by its fundamental component. The load current may be approximated by

$$i_{o1} \approx I_{o1}\sin(\omega t - \theta) \tag{9.31}$$

where

$$\theta = \tan^{-1}\frac{\omega L}{R}$$

$$I_{o1} = \frac{V_{o1}}{\sqrt{(\omega L)^2+R^2}}$$

$$V_{o1} = \frac{4V_{dc}}{\pi}$$

The average power delivered to the load is given by

$$P_{\text{ave}} = I_{o1,\text{rms}}V_{o1,\text{rms}}\cos\theta$$

$$= \frac{I_{o1}V_{o1}}{2}\cos\theta$$

$$P_{\text{ave}} = \frac{8V_{dc}^2 R}{\pi^2\sqrt{(\omega L)^2+R^2}}\cos\theta \tag{9.32}$$

where $I_{o1,\text{rms}}$ is the rms of the fundamental component of the inductor current.

Including the contribution of higher harmonics to the power delivered to the load leads to more complex equations for the power factor and total harmonic distortion.

**EXERCISE 9.3**

Consider the full-bridge voltage-source inverter under an $R$-$L$ load of Fig. 9.17($a$) with $V_{dc} = 220$ V, $L = 6$ mH, $R = 16\ \Omega$, and $f_s = 50$ Hz. Calculate the average power delivered to the load up to the seventh harmonic.

***ANSWER*** 2.73 W

### 9.3.2 Generalized Analysis

As in the half-bridge configuration, the preceding analysis under 50% duty cycle control with no $\alpha$ overlap does not allow output control. Figure 9.18($a$) shows a typical output voltage under $\alpha$ control that is produced using the switching sequence of Fig. 9.14($a$). The equivalent circuit for the single-phase bridge inverter is shown in Fig. 9.18($b$).

Let us assume we have a generalized impedance load, $Z_{load}$, as shown in Fig. 9.19. The Fourier analysis representation for $v_o(t)$ is given by

$$v_o(t) = \sum_{n=1, 3, 5, \ldots}^{\infty} V_n \sin n\omega t \tag{9.33}$$

where

$$V_n = \frac{4V_{dc}}{n\pi} \cos n\alpha \tag{9.34}$$

and $i_o(t)$ is obtained from

$$i_o(t) = \sum_{n=1,3,5,\ldots}^{\infty} I_n \sin(n\omega t - \theta_n) \tag{9.35}$$

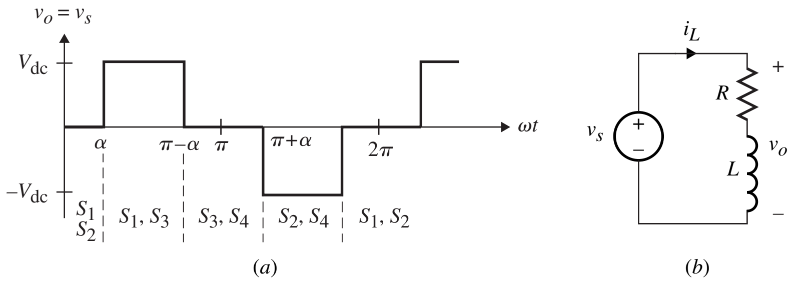



**Figure 9.18** ($a$) Output voltage using switching sequence given in Fig. 9.14($a$). ($b$) Equivalent circuit for the full-bridge inverter.

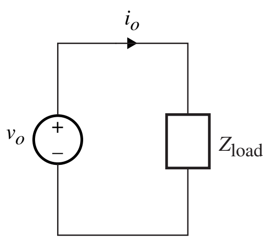



**Figure 9.19** Generalized load representation.

where

$$I_n = \frac{V_n}{|Z_n|} = \frac{4V_{dc}}{n\pi|Z_n|}\cos n\alpha \tag{9.36a}$$

$$\theta_n = \angle Z_n \tag{9.36b}$$

where $|Z_n|$ is the magnitude of the $n$th harmonic impedance, and $\angle Z_n$ is the angle of the $n$th harmonic impedance. The overall rms output voltage is given by

$$V_{o,\text{rms}} = V_{dc}\sqrt{1 - \frac{2\alpha}{\pi}} \tag{9.37}$$

In terms of the $n$th harmonic, the rms value of the voltage for each harmonic is given by

$$V_{on,\text{rms}} = \frac{4V_{dc}}{n\sqrt{2}\pi}\sqrt{\sum_{n=1,3,5,\ldots}^{\infty} \cos^2 n\alpha} \tag{9.38}$$

and the rms value of the current for each harmonic is given by

$$I_{on,\text{rms}} = \sqrt{\sum_{n=1,3,5,\ldots}^{\infty} I_{n,\text{rms}}^2} \tag{9.39}$$

The rms for the $n$th harmonic is given by

$$I_{n,\text{rms}} = \frac{4V_{dc}}{\sqrt{2}n\pi|Z_n|}\cos n\alpha \tag{9.40}$$

and the rms of the fundamental output current is given by

$$I_{o1,\text{rms}} = \frac{4V_{dc}}{\sqrt{2}\pi|Z_1|}\cos\alpha \tag{9.41}$$

where

$$|Z_1| = \sqrt{R^2 + (\omega_0 L)^2}$$

and $\omega_0$ is the fundamental frequency.

The total average power delivered to the load resistance is given by

$$P_{o,\text{ave}} = \frac{1}{T}\int_0^T i_o v_o \, dt \tag{9.42}$$

The voltage and current THDs are given in Eqs. (9.43$a$) and (9.43$b$), respectively.

$$\begin{aligned} \text{THD}_v &= \sqrt{\left(\frac{V_{o,\text{rms}}}{V_{o1,\text{rms}}}\right)^2 - 1} \\ &= \frac{\pi}{2\cos\alpha}\sqrt{\frac{1}{2} - \frac{\alpha}{\pi}} \end{aligned} \tag{9.43a}$$

$$\text{THD}_i = \sqrt{\left(\frac{I_{o,\text{rms}}}{I_{o1,\text{rms}}}\right)^2 - 1} \tag{9.43b}$$

$$= \sqrt{\left(\sum_{n=1,3,5,\ldots}^{\infty} \frac{|Z_1|}{n|Z_n|}\frac{\cos\alpha}{\cos n\alpha}\right)^2 - 1}$$

For the $n$th harmonic, the average power of Eq. (9.42) is given by

$$P_{on,\text{ave}} = I_{on,\text{rms}} V_{on,\text{rms}} \cos\theta_n$$

So the total average power is given by

$$P_{o,\text{ave}} = \sum_{n=1,3,5,\ldots}^{\infty} I_{on,\text{rms}} V_{on,\text{rms}} \cos\theta_n \tag{9.44}$$

---

## EXAMPLE 9.5

Consider the full-bridge inverter whose equivalent circuit is represented in Fig. 9.19 with the four different loads shown in Fig. 9.20 with $R = 8\ \Omega$, $L = 30\ \mu\text{H}$, $C = 147\ \text{mF}$, $f_o = 60\ \text{Hz}$, $V_{dc} = 120\ \text{V}$, and $\alpha = \pi/6$.

**(a)** Determine the rms for $i_o$ and $v_o$ for the first, third, fifth, and seventh harmonics.

**(b)** Determine the total average power delivered to the load for each of the above harmonics.

**(c)** Determine the output current and voltage total harmonic distortion.

***SOLUTION*** **(a)** To determine the rms for $i_o$ and $v_o$ for the first, third, fifth, and seventh harmonics, we use the $n$th harmonic, given by

$$v_{n,o} = \frac{4V_{dc}}{n\pi}\cos n\alpha\ \sin n\omega t \tag{9.45}$$

$$I_{n,\text{rms}} = \frac{4V_{dc}}{\sqrt{2}n\pi|Z_n|}\cos n\alpha$$

*(i) For the resistive load:*

$$V_{n,\text{rms}} = \frac{2\sqrt{2}V_{dc}}{n\pi}\cos n\alpha \qquad I_{n,\text{rms}} = \frac{V_{n,\text{rms}}}{|Z_n|},\ |Z_n| = R = 8\ \Omega,\ \theta_n = 0,\ \alpha = \pi/6\,.$$

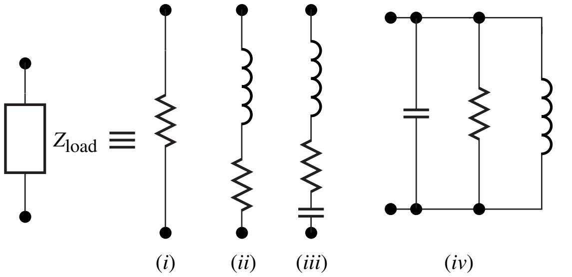



**Figure 9.20** Various types of loads.

$$V_{1,\text{rms}} = \frac{2\sqrt{2}\cdot 120}{\pi}\cos\left(\frac{\pi}{6}\right) = 93.56 \text{ V} \qquad I_{1,\text{rms}} = \frac{V_{1,\text{rms}}}{|Z_1|} = \frac{V_{1,\text{rms}}}{R} = 11.69 \text{ A}$$

$$V_{3,\text{rms}} = \frac{2\sqrt{2}\cdot 120}{3\pi}\cos\left(\frac{\pi}{2}\right) = 0 \text{ V} \qquad I_{3,\text{rms}} = \frac{V_{3,\text{rms}}}{|Z_3|} = \frac{V_{3,\text{rms}}}{R} = 0 \text{ A}$$

$$V_{5,\text{rms}} = \frac{2\sqrt{2}\cdot 120}{5\pi}\cos\left(\frac{5\pi}{6}\right) = -18.71 \text{ V} \qquad I_{5,\text{rms}} = \frac{V_{5,\text{rms}}}{|Z_5|} = \frac{V_{5,\text{rms}}}{R} = -2.34 \text{ A}$$

$$V_{7,\text{rms}} = \frac{2\sqrt{2}\cdot 120}{7\pi}\cos\left(\frac{7\pi}{6}\right) = -13.37 \text{ V} \qquad I_{7,\text{rms}} = \frac{V_{7,\text{rms}}}{|Z_7|} = \frac{V_{7,\text{rms}}}{R} = -1.67 \text{ A}$$

Using the first four harmonics, the approximate rms values for $v_o$ and $i_o$ are given by

$$V_{o,\text{rms}} = \sqrt{V_{1,\text{rms}}^2 + V_{3,\text{rms}}^2 + V_{5,\text{rms}}^2 + V_{7,\text{rms}}^2} = 96.34 \text{ V}$$

$$I_{o,\text{rms}} = \sqrt{I_{1,\text{rms}}^2 + I_{3,\text{rms}}^2 + I_{5,\text{rms}}^2 + I_{7,\text{rms}}^2} = 12.04 \text{ A}$$

The exact rms values are given by

$$V_{o,\text{rms}} = \sqrt{\frac{2}{2\pi}\int_{\alpha}^{\pi} V_{\text{dc}}^2 \, d\omega t} = \sqrt{\frac{5}{6}}V_{\text{dc}} = 109.59 \text{ V}$$

$$I_{o,\text{rms}} = \frac{V_{o,\text{rms}}}{|Z_n|} = 13.69 \text{ A}$$

*(ii) For the series RL load:* The rms value for the output voltage is the same as above, since the output voltage is independent of the load.

$$I_{n,\text{rms}} = \frac{V_{n,\text{rms}}}{|Z_n|}, \qquad |Z_n| = \sqrt{R^2 + (n\omega L)^2}, \qquad \theta_n = \tan^{-1}\left(\frac{n\omega L}{R}\right)$$

$$I_{1,\text{rms}} = \frac{V_{1,\text{rms}}}{\sqrt{R^2 + (\omega L)^2}} = 6.75 \text{ A}$$

$$I_{3,\text{rms}} = \frac{V_{3,\text{rms}}}{\sqrt{R^2 + (3\omega L)^2}} = 0 \text{ A}$$

$$I_{5,\text{rms}} = \frac{V_{5,\text{rms}}}{\sqrt{R^2 + (5\omega L)^2}} = -0.33 \text{ A}$$

$$I_{7,\text{rms}} = \frac{V_{7,\text{rms}}}{\sqrt{R^2 + (7\omega L)^2}} = -0.168 \text{ A}$$

*(iii) For the series RLC load:* The rms value for the output voltage is the same as in *(i)*, since the output voltage is independent of the load.

$$|Z_n| = \sqrt{R^2 + \left(n\omega L - \frac{1}{n\omega C}\right)^2}, \quad \theta_n = \tan^{-1}\left(\frac{n\omega L - \frac{1}{n\omega C}}{R}\right)$$

$$I_{1,\text{rms}} = \frac{V_{1,\text{rms}}}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} = 8.95 \text{ A}$$

$$I_{3,\text{rms}} = \frac{V_{3,\text{rms}}}{\sqrt{R^2 + \left(3\omega L - \dfrac{1}{3\omega C}\right)^2}} = 0 \text{ A}$$

$$I_{5,\text{rms}} = \frac{V_{5,\text{rms}}}{\sqrt{R^2 + \left(5\omega L - \dfrac{1}{5\omega C}\right)^2}} = -0.35 \text{ A}$$

$$I_{7,\text{rms}} = \frac{V_{7,\text{rms}}}{\sqrt{R^2 + \left(7\omega L - \dfrac{1}{7\omega C}\right)^2}} = -0.17 \text{ A}$$

*(iv) For the parallel RLC load:* The rms value for the output voltage is the same as the one shown above, since the output voltage is independent of the load.

$$|Z_n| = \frac{1}{\sqrt{\left(\dfrac{1}{R}\right)^2 + \left(\dfrac{n^2\omega^2 LC - 1}{n\omega L}\right)^2}}, \qquad \theta_n = -\tan^{-1}\left(\frac{n^2\omega^2 LRC - R}{n\omega L}\right)$$

$$I_{1,\text{rms}} = \frac{V_{1,\text{rms}}}{1 \Big/ \sqrt{\left(\dfrac{1}{R}\right)^2 + \left(\dfrac{1^2\omega^2 LC - 1}{\omega L}\right)^2}} = 12.1 \text{ A}$$

$$I_{3,\text{rms}} = \frac{V_{3,\text{rms}}}{1 \Big/ \sqrt{\left(\dfrac{1}{R}\right)^2 + \left(\dfrac{3^2\omega^2 LC - 1}{3\omega L}\right)^2}} = 0 \text{ A}$$

$$I_{5,\text{rms}} = \frac{V_{5,\text{rms}}}{1 \Big/ \sqrt{\left(\dfrac{1}{R}\right)^2 + \left(\dfrac{5^2\omega^2 LC - 1}{5\omega L}\right)^2}} = -5.39 \text{ A}$$

$$I_{7,\text{rms}} = \frac{V_{7,\text{rms}}}{1 \Big/ \sqrt{\left(\dfrac{1}{R}\right)^2 + \left(\dfrac{7^2\omega^2 LC - 1}{7\omega L}\right)^2}} = -5.29 \text{ A}$$

**(b)** Power calculations are obtained from the following equation:

$$P_{on,\text{ave}} = I_{on,\text{rms}} V_{on,\text{rms}} \cos\theta_n$$

*(i) For the resistive load:*

$$\cos\theta_n = 1 \quad \text{for } n = 1, 3, 5, \ldots$$

$$P_{o1,\text{ave}} = (93.56)(11.69) = 1094.65 \text{ W}$$

$$P_{o3,\text{ave}} = 0 \text{ W}$$

$$P_{o5,\text{ave}} = (-18.71)(-2.34) = 43.78 \text{ W}$$

$$P_{o7,\text{ave}} = (-13.37)(-1.67) = 22.33 \text{ W}$$

$$P_o = \sum_{n=1,3,5,\ldots}^{\infty} I_{on,\text{rms}} V_{on,\text{rms}} \cos\theta_n = 1160.76 \text{ W}$$

*(ii) For the series RL load:*

$$P_{o1,\text{ave}} = (93.56)(6.75)\cos(54.73°) = 364.67 \text{ W}$$
$$P_{o3,\text{ave}} = 0 \text{ W}$$
$$P_{o5,\text{ave}} = (-18.71)(-0.33)\cos(81.95°) = 0.865 \text{ W}$$
$$P_{o7,\text{ave}} = (-13.37)(-0.168)\cos(84.23°) = 0.226 \text{ W}$$
$$P_{o,\text{ave}} = \sum_{n=1,3,5,\ldots}^{\infty} I_{on,\text{rms}} V_{on,\text{rms}} \cos\theta_n = 365.75 \text{ W}$$

*(iii) For the series RLC load:*

$$P_{o1,\text{ave}} = (93.56)(9.95)\cos(-40.093°) = 640.58 \text{ W}$$
$$P_{o3,\text{ave}} = 0 \text{ W}$$
$$P_{o5,\text{ave}} = (-18.71)(-0.35)\cos(81.36°) = 0.98 \text{ W}$$
$$P_{o7,\text{ave}} = (-13.37)(-0.17)\cos(84.037°) = 0.24 \text{ W}$$
$$P_{o,\text{ave}} = \sum_{n=1,3,5,\ldots}^{\infty} I_{on,\text{rms}} V_{on,\text{rms}} \cos\theta_n = 641.8 \text{ W}$$

*(iv) For the parallel RLC load:*

$$P_{o1,\text{ave}} = (93.56)(12.1)\cos(-14.76°) = 1094.2 \text{ W}$$
$$P_{o3,\text{ave}} = 0 \text{ W}$$
$$P_{o5,\text{ave}} = (-18.17)(-5.39)\cos(-64.274°) = 43.78 \text{ W}$$
$$P_{o7,\text{ave}} = (-13.37)(-5.29)\cos(-71.579°) = 22.35 \text{ W}$$
$$P_{o,\text{ave}} = \sum_{n=1,3,5,\ldots}^{\infty} I_{on,\text{rms}} V_{on,\text{rms}} \cos\theta_n = 1160.3 \text{ W}$$

**(c)** The total harmonic distortion can be obtained from Eq. (9.43):

$$\text{THD}_i = \sqrt{\left(\frac{I_{o,\text{rms}}}{I_{o1,\text{rms}}}\right)^2 - 1} = 0.31$$

---

## EXERCISE 9.4

Consider the inductive-resistive–load inverter shown in Fig. 9.17($a$) with the following parameters: $\alpha = \pi/3$, $L = 100$ mH, $R = 16\ \Omega$, $V_{dc} = 250$ V, and $f_o = 60$ Hz. Determine the fundamental current and voltage components.

***ANSWER*** $i_{o1} = \dfrac{12.2}{\pi}\sin(\omega t - 67°)\ \text{A}$

The general analysis presented in this section is based on a passive load $Z_L$ that produces a fixed load phase shift $\angle Z_L = \theta$; therefore, the output voltage control can be achieved only by varying $\alpha$. However, when the load contains a voltage source, as in motor and utility grid applications, then it is an active load that allows both the phase shift $\theta$ and the switching displacement $\alpha$ to be used as control variables.

## EXAMPLE 9.6

Consider the active load in a bridge inverter that consists of an *R*-*L* load and an ac sinusoidal voltage source as shown in Fig. 9.21 with the load voltage $v_o$ as shown. Obtain the expression for the fundamental load current and the average power delivered to the load for the following circuit parameters: $v_{ac} = 100\sin(2\pi 100t - 30°)$, $V_{dc} = 180$ V, $L = 42$ mH, $R = 0.5\ \Omega$, $\alpha = 15°$, and $f_o = 60$ Hz.

***SOLUTION*** To obtain the expression for the fundamental load current, we can use the following equations (the arrow indicates phasor notation):

$$\overrightarrow{I_{o1}} = \frac{\overrightarrow{V_{o1}} - \overrightarrow{V_{ac}}}{R + j\omega L}$$

We have

$$V_{o1} = \frac{4V_{dc}}{\pi}\cos\alpha = \frac{4\times 180}{\pi}\cos(15°) = 221.37\ \text{V}$$

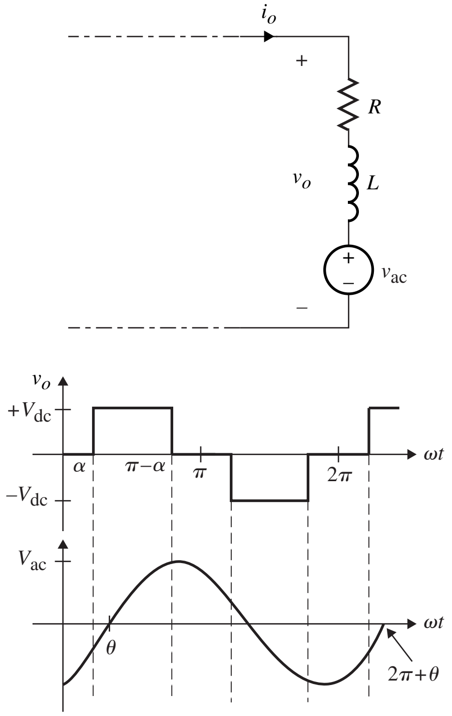



**Figure 9.21** Inverter example with an active load.

Hence,

$$\overrightarrow{V_{o1}} = 221.37\angle 0° \text{ V}$$

$$\overrightarrow{V_{\text{ac}}} = 100\angle 30° \text{ V}$$

$$\overrightarrow{I_{o1}} = \frac{221.37\angle 0° \text{ V} - 100\angle 30°}{(0.5 + j2\pi(60))(42 \times 10^{-3})} = 9.08\angle -110.25° \text{ A}$$

The total power delivered to the load is given by the following equation:

$$\overrightarrow{P_T} = \frac{1}{2}\overrightarrow{I_{o1}}\overrightarrow{V_{o1}}$$

$$\overrightarrow{P_T} = \frac{1}{2}(221.37\angle 0°)(9.08\angle -110.25°) = 1005.02\angle -110.25° \text{ W}$$

## 9.4 HARMONIC REDUCTION

Harmonic reduction includes the elimination and cancellation of certain harmonics of the output voltage. Reducing the harmonic content of the ac output is one of the most difficult challenges of dc-ac inverter design. If the inverter drives an electromechanical load (ac motor), the harmonics can excite a mechanical resonance, causing the load to emit acoustic noise.

Unlike the case in dc-to-dc converters, harmonics in the output waveforms are very significant. This is why the output filters perform different functions and their design is quite different in terms of complexity and size. In designing output filters in dc-to-dc converters, the objective is to limit the output voltage ripple to a certain desired percentage of the average output voltage. In other words, this is a passive filter approach that is limited by the physical size of inductors and capacitors. In dc-to-dc inverters, the reduction or cancellation of the output harmonics is done actively by controlling the switching technique of the inverter. Compared with ac-to-dc conversion, harmonic filtering in dc-to-ac is harder since it will affect the attenuation and/or the phase shift of the fundamental component.

The harmonics that are present in the inverter's output voltage are high for many practical applications, especially when the output voltage needs to be near sinusoidal. To help produce a near sinusoidal output, electronic low-pass filters are normally added at the output to remove third and higher harmonics. The design of such filters tends to be challenging when the third and fifth harmonics close to the fundamental are high. In most cases, effective filters require large size and high numbers of filtering capacitive and inductive components, resulting in bulkiness. By controlling the width or the number of pulses, certain harmonic content can be removed without the need for complex harmonic filtering circuits.

It is possible to cancel certain harmonics by simply selecting the duration of the pulse in the half-cycle of the output voltage. Recall that the peak component of the *n*th harmonic for Fig. 9.18(*a*) is given by

$$V_n = \frac{4V_{\text{dc}}}{n\pi}\cos n\alpha \tag{9.46}$$

For the third harmonic, we have

$$V_3 = \frac{4V_{\text{dc}}}{3\pi}\cos 3\alpha \tag{9.47}$$

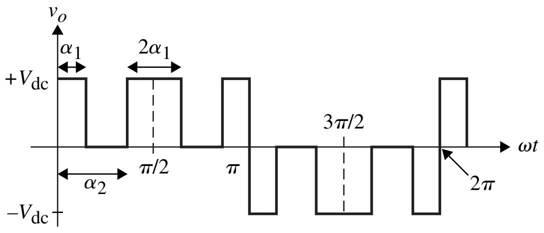



**Figure 9.22** Two-angle shift control of inverter output voltage.

To cancel the third harmonic, we set $\alpha = \pi/6$; this results in the cancellation of all harmonics of the order of $3n$.

Consider the case of a two-pulse output of Fig. 9.22 with angles of $\alpha_1$ and $\alpha_2$ as shown. It can be shown that the $n$th harmonic is given by

$$V_{on} = \frac{4V_{dc}}{n\pi}(1 - \cos n\alpha_1 + \cos n\alpha_2) \tag{9.48}$$

For example, to eliminate the third and fifth harmonics, we make $V_{on} = 0$ at two frequencies as follows:

$$1 - \cos 3\alpha_1 + \cos 3\alpha_2 = 0$$

$$1 - \cos 5\alpha_1 + \cos 5\alpha_2 = 0$$

Solve for $\alpha_1$ and $\alpha_2$ to obtain $\alpha_1 = 17.8°$ and $\alpha_2 = 38°$.

## EXAMPLE 9.7

Consider two cascaded push-pull inverters as shown in Fig. 9.23($a$); the switching waveforms for $S_1 - S_4$ are shown in Fig. 9.23($b$). Sketch the waveforms for the outputs $v_{o1}$, $v_{o2}$, and $v_o$.

***SOLUTION*** The waveforms for the output voltage are shown in Fig. 9.23($b$).

The fundamental component of $v_o$ is given by

$$v_{o1}(t) = \frac{8V_{dc}}{\pi}\cos\alpha \sin\omega t$$

where $\alpha = \omega t_1$. It is clear that the magnitude of the fundamental component of $v_{o1}$, has been reduced by $\cos\alpha$ compared to the case when no phase is present ($\alpha = 0$).

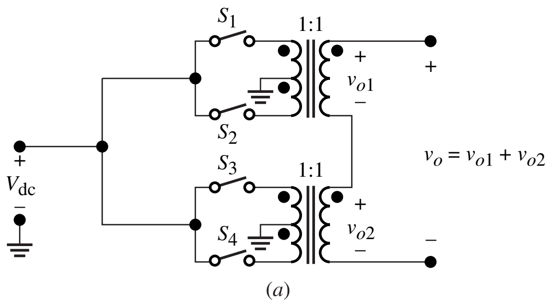
### Harmonic Analysis

The harmonics content present in the output of the inverter could be significant. Depending on the application, reducing the effect of these harmonics can be very important. Recall that the $n$th component for a squarewave output ($\alpha = 0$) is $4V_{dc}/n\pi$. For $\alpha \neq 0$, the expression for $v_o(t)$ is given by

$$v_o(t) = \sum_{n=1,3,5,\ldots} V_n \sin\omega t \tag{9.49}$$

where

$$V_n = \frac{4V_{dc}}{n\pi}\cos n\alpha$$



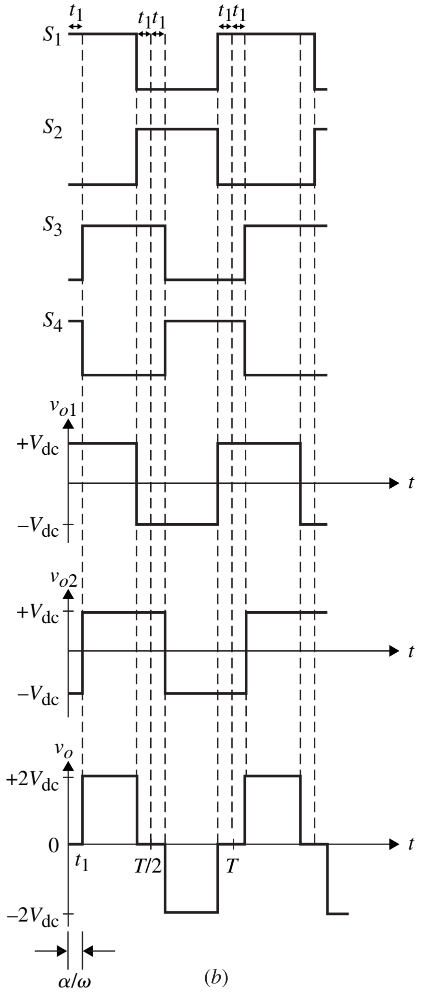
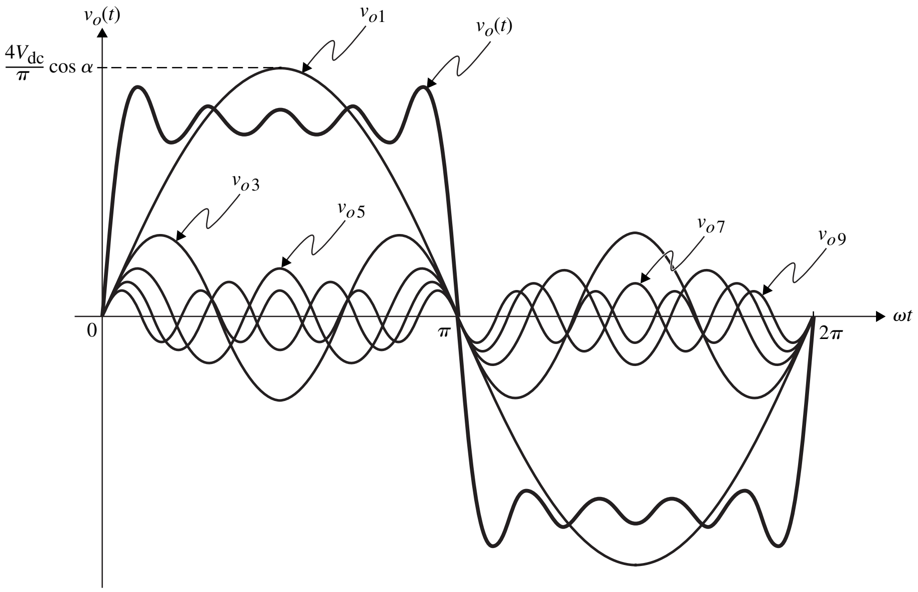




**Figure 9.23** (*a*) Two push-pull inverters for Example 9.7. (*b*) Typical switching waveforms.

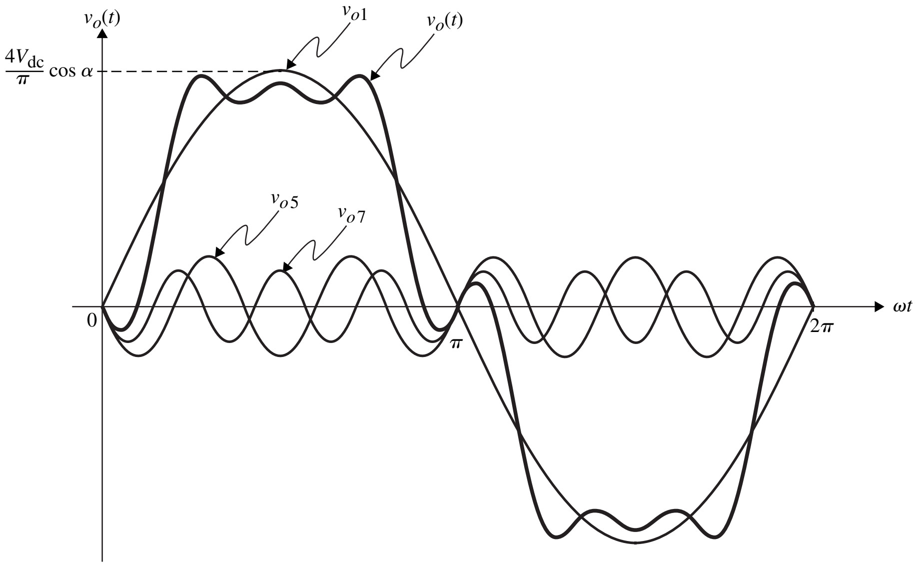
The fundamental output voltage component is given by

$$v_{o1} = \frac{4V_{dc}}{\pi}\cos\alpha\sin\omega t \tag{9.50}$$

Figure 9.24(*a*), (*b*), and (*c*) shows the resultant output voltage where harmonics up to the ninth are included for $\alpha = 0$, $\alpha = \pi/6$, and $\alpha = \pi/3$, respectively. The sketch of the magnitude of the harmonics of Fig. 9.24, as a function of $\alpha$, is shown in Fig. 9.25.




(*a*)



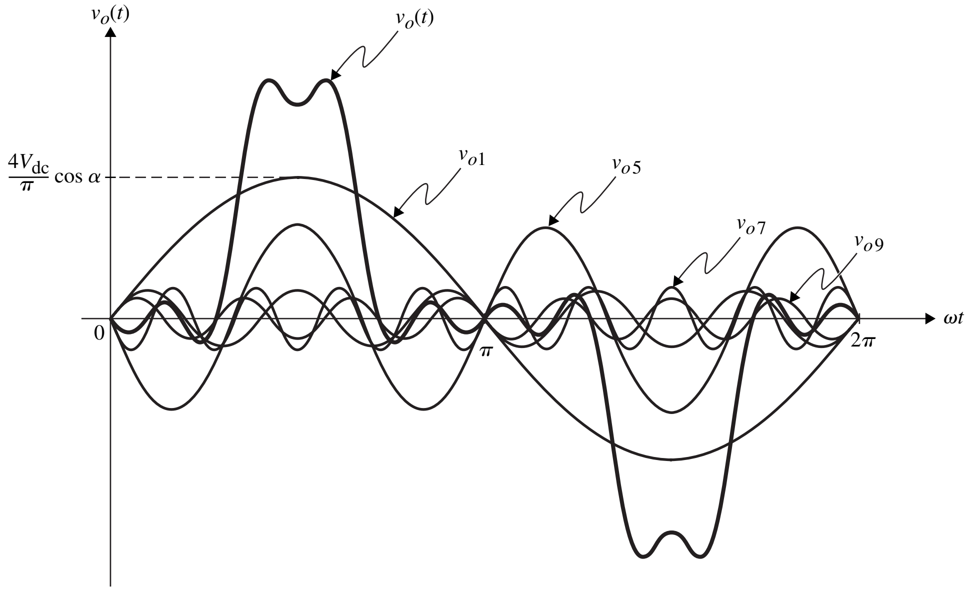

(*b*)

**Figure 9.24** The first nine output harmonics: (*a*) $\alpha = 0$ . (*b*) $\alpha = \pi/6$ .

It is clear from the earlier equation that the magnitude of the harmonics is inversely proportional to the magnitude of *n* and $\alpha$. Under a wide range of variation of $\alpha$, the filtering of the harmonics might not be an easy task.

The total harmonic distortion for $v_o(t)$ is given by

$$\text{THD}_V = \sqrt{\left(\frac{V_{o,\text{rms}}}{V_{o1,\text{rms}}}\right)^2 - 1} \tag{9.51}$$




(*c*)

**Figure 9.24** (*continued*) (*c*) $\alpha = \pi/3$.

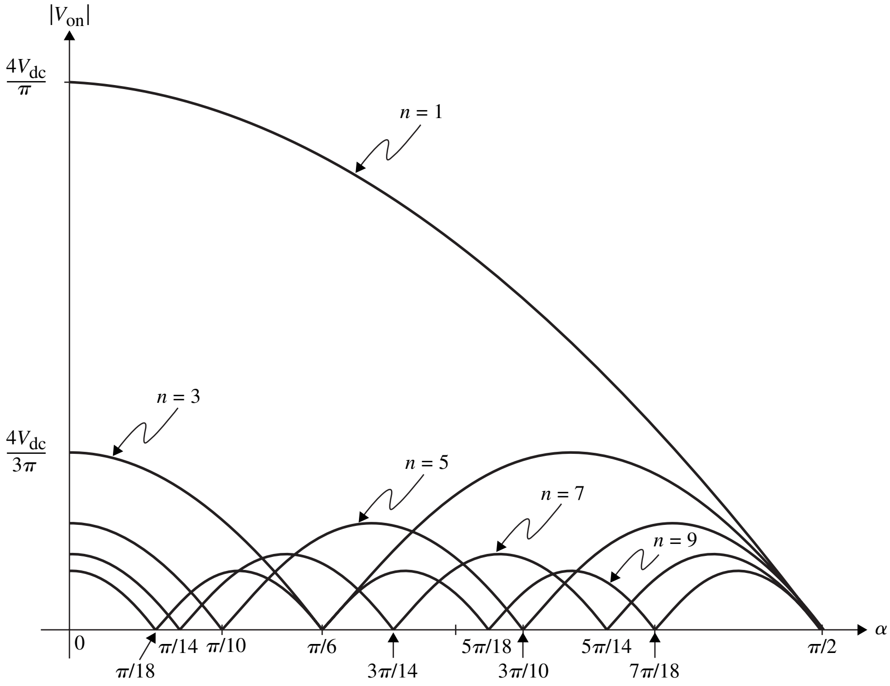



**Figure 9.25** Magnitude of the output harmonics as a function of $\alpha$.

$$V_{o,\text{rms}}^2 = V_{o1,\text{rms}}^2 + V_{o2,\text{rms}}^2 + \cdots + V_{on,\text{rms}}^2$$
$$= \sum_{n=1,3,5,\ldots} V_{on,\text{rms}}^2 \tag{9.52}$$

where

$$V_{on,\text{rms}} = \frac{V_{on}}{\sqrt{2}} \tag{9.53a}$$

$$V_{on} = \frac{4V_{\text{dc}}}{n\pi}\cos n\alpha \tag{9.53b}$$

Substitute Eqs. (9.52) and (9.53) into Eq. (9.51) to yield

$$\text{THD}_V = \sqrt{\left(\frac{\sum_{n=1,3,5,\ldots} V_{on,\text{rms}}^2}{V_{o1,\text{rms}}^2}\right) - 1}$$

$$\text{THD}_V = \sqrt{\left(\frac{\sum_{n=3,5,7,\ldots} V_{on,\text{rms}}^2}{V_{o1,\text{rms}}^2}\right)} \tag{9.54}$$

$$\text{THD}_V = \frac{1}{\cos\alpha}\sqrt{\sum_{n=3,5,\ldots}\frac{1}{n^2}\cos^2 n\alpha}$$

The plots of the THD for $\alpha = 0$ to $\alpha = \pi/2$ are shown in Fig. 9.26.

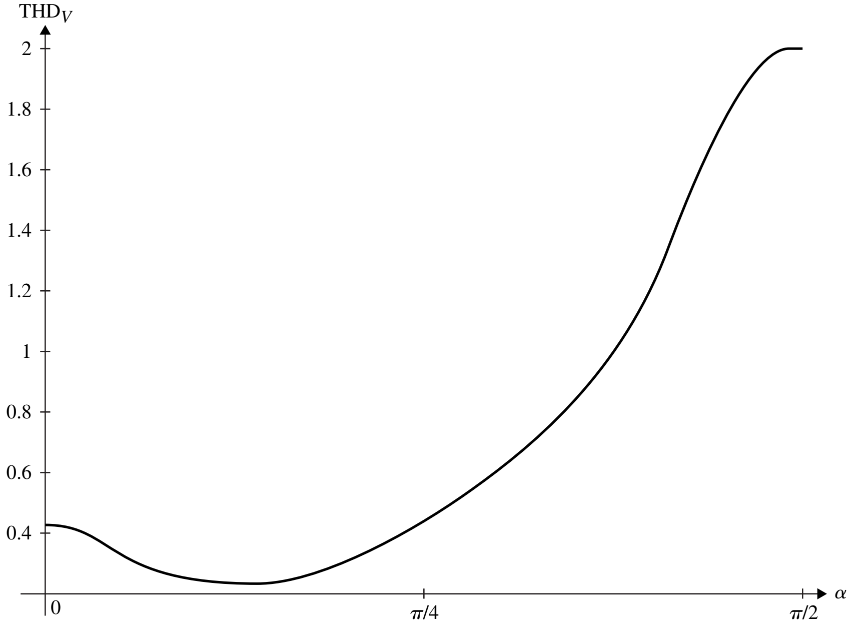



**Figure 9.26** Plots of THD as a function of $\alpha$.

**EXERCISE 9.5**

Show that the ratio of the harmonic magnitude with respect to its fundamental component for Fig. E9.5 is given by

$$\frac{V_{on}}{V_{o1}} = \frac{\sin(n\theta/2)}{n\sin(\theta/2)}$$

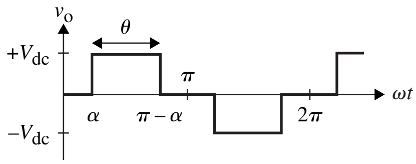



**Figure E9.5** Inverter voltage with $\alpha$ control.

**EXERCISE 9.6**

Determine the value of $\alpha > 0$ that produces the largest THD in Eq. (9.54) when only the third harmonics are included.

***ANSWER*** $\alpha = \pi/2$

## 9.5 PULSE-WIDTH MODULATION

Figure 9.27 shows the simplified block diagram representation for a single-phase switching-mode inverter. The output $v_o'(t)$ shows possible types of output waveforms that can be produced depending on the pulse-width modulation (PWM) control technique employed.

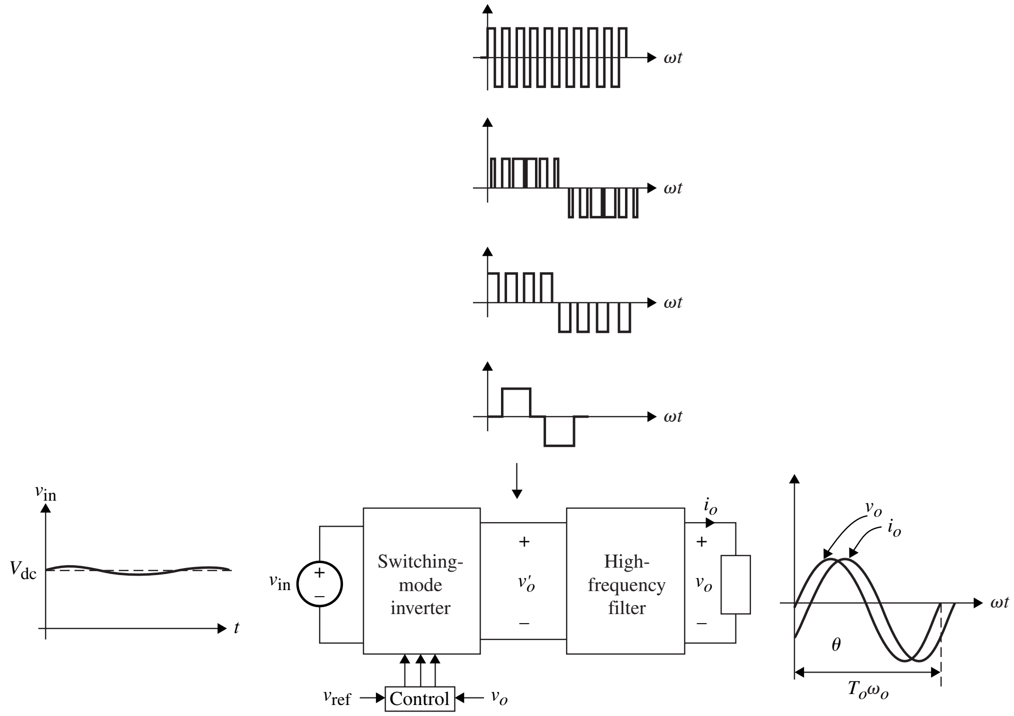



**Figure 9.27** Simplified block diagram of single-phase switching-mode inverter.

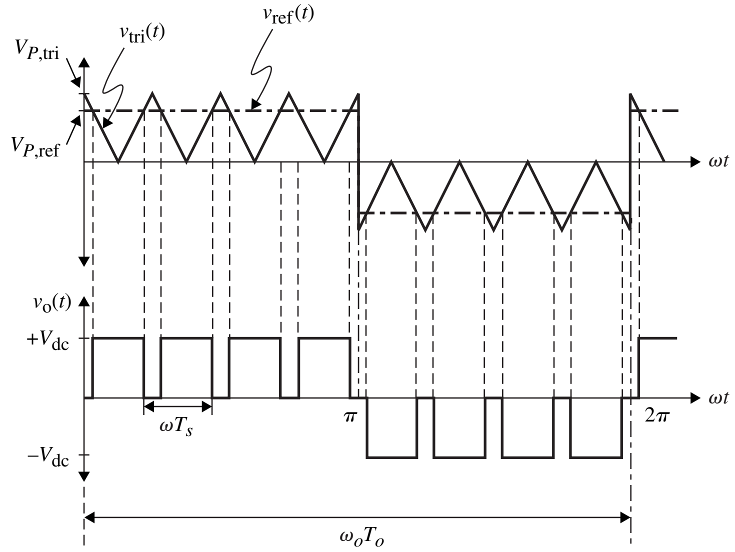



**Figure 9.28** Typical waveform for equal-pulse PWM technique.

In the single-phase inverter, one or more pulses in a given half-cycle are used to control the output voltage. Since varying the width of these pulses within the half-cycle carries out the control, the process is appropriately known as pulse-width modulation (PWM). By modulating the width of several pulses per half-cycle, a more efficient method of controlling the output voltage of the inverter is obtained. Using the PWM process, we can extract a low-frequency signal from a train of high-frequency squarewaves.

Generally speaking, the PWM method may be divided into two classes, depending on the modulation technique:

1. Nonsinusoidal PWM, in which all pulses have the same width and are normally modulated equally to control the output voltage, as shown in the four-pulse-per-half-cycle example in Fig. 9.28. The widths of these pulses are adjusted equally to control the output voltage. Eliminating a selected number of series harmonics as discussed before requires a very complex control technique to generate the required switching sequence.
2. Sinusoidal PWM, which allows the pulse width to be modulated sinusoidally; i.e., the width of each pulse is proportional to the instantaneous value of a reference sinusoid whose frequency equals that of the fundamental components as shown in the six-pulse-per-half-cycle example in Fig. 9.29.

Normally the pulses in the nonsinusoidal PWM method are arranged to produce an odd-function output voltage that is symmetrical around $T_o/2$. This results in the cancellation of all the cosine terms and the even harmonics of the sine terms.

We notice that the reference voltage $v_{ref}(t)$ is square and sinusoidal for the equal-pulse and sinusoidal PWM methods, respectively. It is important that we first define some terms that pertain to the square and sinusoidal PWM inverters that will be discussed.

$v_{tri}(t)$: Repetitive triangular waveform (also known as a carrier signal)

$V_{P,\,tri}$: Peak value of the triangular waveform

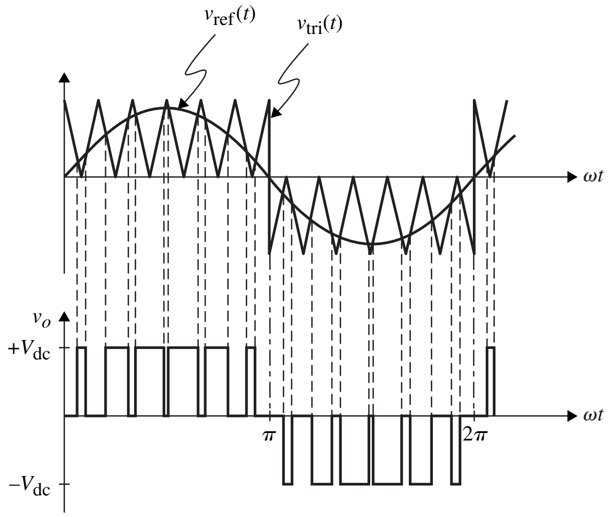



**Figure 9.29** Typical waveforms for sinusoidal PWM technique.

| | |
|---|---|
| $T_s, f_s$ : | Period and frequency of the triangular waveform ($f_s$ is also known as the carrier or switching frequency) |
| $v_{ref}$ : | Reference signal, which can be either a square or a sinusoidal waveform (also known as a control signal) |
| $V_{P,\,ref}$ : | Peak value of the reference signal |
| $T_o, f_o$ : | Desired inverter output period and output frequency, which are equal to the period and frequency of the reference or control signal |
| $m_a$ : | Inverter amplitude modulation index |
| $m_f$: | Inverter frequency modulation index |
| $k$: | Number of pulses per half-cycle |

The amplitude and frequency modulation indices are defined as follows:

$$m_a = \frac{V_{P,\text{ref}}}{V_{P,\text{tri}}} \tag{9.55}$$

$$m_f = \frac{f_s}{f_o} \tag{9.56}$$

Next we discuss the two well-known PWM techniques.

### 9.5.1 Equal-Pulse (Uniform) PWM

The equal-pulse PWM technique, known also as single-pulse PWM control, is very old and less popular nowadays. The technique is very simple and requires simple control since all generated pulses have equal widths. Generating the equal and multiple pulses is achieved by comparing a square reference voltage waveform $v_{ref}(t)$ to a triangular control (carrier) voltage waveform, $v_{cont}(t)$, as shown in Fig. 9.28. The op-amp produces a triggering signal every time the carrier signal goes below or above the reference signal, as shown in the figure. It is clear that the frequency of the reference voltage waveform determines the frequency of the output voltage, and the frequency of the control signal determines the number of equal pulses in each half-cycle.

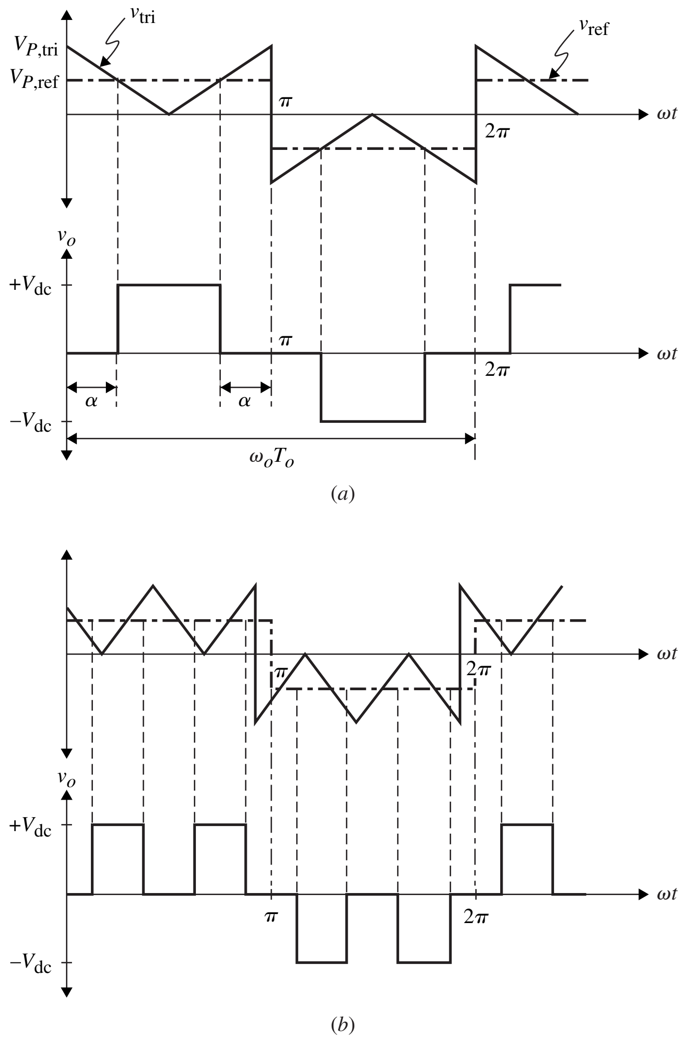



**Figure 9.30** Examples of equal pulses. (*a*) One-pulse output. (*b*) Two-pulse output.

Figure 9.30(*a*), (*b*), (*c*), and (*d*) shows examples of one-, two-, three-, and seven-pulse outputs, respectively. As the magnitude of the reference signal increases, the pulse width increases as well, reaching its maximum value of $\pi$ when the magnitude of the reference signal becomes equal to the peak of the modulating signal.

It can be shown the $\alpha$ can be expressed in terms of $V_{P,\text{tri}}$ and $V_{P,\text{ref}}$ as follows:

$$\alpha = -\frac{\pi}{2}\left(\frac{V_{P,\text{ref}}}{V_{P,\text{tri}}} - 1\right) = \frac{\pi}{2}(1 - m_a) \tag{9.57}$$

Notice that the frequency of the control signal, $f_{\text{tri}}$, is twice the frequency of the reference signal, $f_{\text{ref}}$. It is clear from these waveforms that the number of pulses, $k$, is equal to the number of periods of the control signal per half-period of the reference signal; i.e., $k$ is the number of switching periods, $T_s$, in the $T_o/2$ period, which can be expressed as

$$k = \frac{1}{2}\frac{f_s}{f_o} \tag{9.58}$$

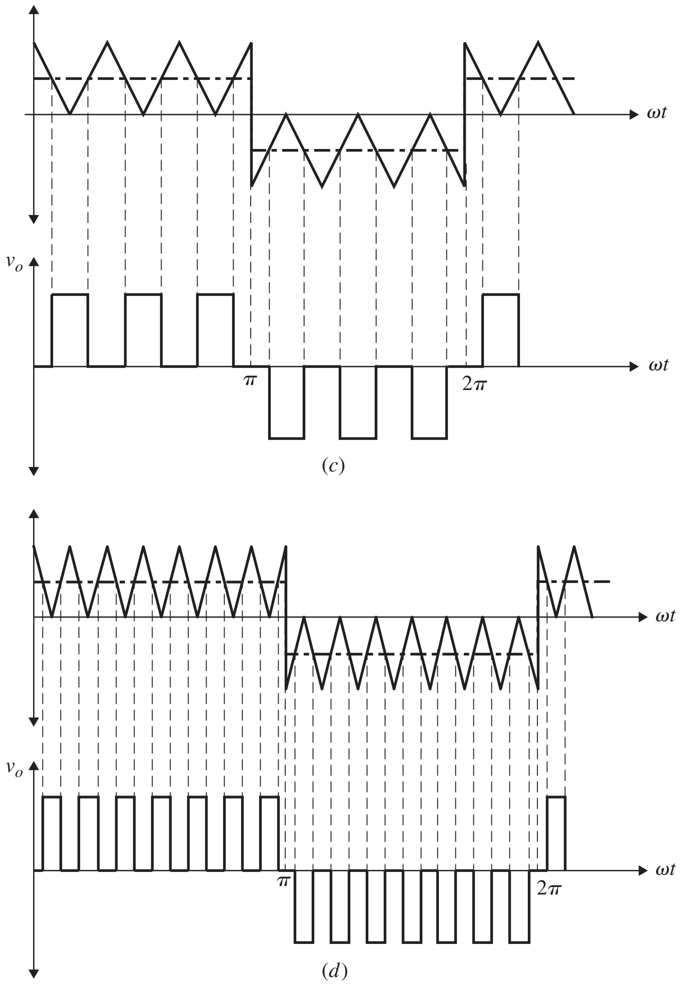



**Figure 9.30** (*continued*) Examples of equal pulses. (*c*) Three-pulse output. (*d*) Seven-pulse output.

In terms of the frequency modulation index, $m_f$, $k$ may be expressed by the following relation:

$$k = \frac{1}{2}m_f \tag{9.59}$$

Notice that $k = 1$ is a special case where $f_s/f_o = 2$.

We should point out that the output frequency is equal to the frequency of the reference signal, (i.e., $f_o = f_{\text{ref}}$) and the switching frequency is equal to the frequency of the carrier or triangle signal ($f_s = f_{\text{tri}}$).

For example, in Fig. 9.30(*d*), $k = 7$ and $m_f = 14$. The maximum width of each pulse occurs when $m_a = 1$ and is given by

$$t_{\text{width,max}} = \frac{T_o}{2k} \tag{9.60}$$

The maximum conduction angle width of each pulse is given by

$$\theta_{\text{width,max}} = \omega_o t_{\text{width,max}} = \frac{\pi}{k} \tag{9.61}$$

**EXERCISE 9.7**

Derive Eq. (9.57).

Next we will derive a general expression for the $i$th pulse width in a given $k$-pulse output in terms of $i$, $k$, and $m_a$. Referring to Fig. 9.31, which shows a $k$-pulse inverter output, the start of the $i$th pulse is given by

$$t_i = (i-1)T_s + \frac{T_s}{2}\left(1 - \frac{V_{P,\text{ref}}}{V_{P,\text{tri}}}\right) \tag{9.62}$$

Substituting for $f_s = 1/T_s$ from Eq. (9.58) into Eq. (9.62), $t_i$ becomes

$$t_i = \frac{T_o}{2k}(i-1) + \frac{T_o}{4k}(1-m_a) \tag{9.63}$$

In terms of the starting angle, $\theta_i = \omega_o t_i$, of the $i$th pulse, Eq. (9.63) may be written as follows:

$$\theta_i = \frac{\pi}{k}\left[i - \frac{m_a}{2} - \frac{1}{2}\right] \tag{9.64}$$

For example, for the two-pulse waveform, $k = 2$ and $m = 0.5$, the angles at which the pulses start are given by

$$\theta_1 = \frac{\pi}{2}\left[1 - \frac{1}{4} - \frac{1}{2}\right] = \frac{\pi}{8} \qquad \theta_3 = \frac{\pi}{2}\left[3 - \frac{1}{4} - \frac{1}{2}\right] = \frac{9\pi}{8}$$

$$\theta_2 = \frac{\pi}{2}\left[2 - \frac{1}{4} - \frac{1}{2}\right] = \frac{5\pi}{8} \qquad \theta_4 = \frac{\pi}{2}\left[4 - \frac{1}{4} - \frac{1}{2}\right] = \frac{13\pi}{8}$$

Notice that because of the symmetry, $\theta_3 = \theta_1 + \pi$ and $\theta_4 = \theta_2 + \pi$. It can be shown that, in general, the width of each pulse is given by

$$\theta_{\text{width}} = \pi\frac{m_a}{k} \tag{9.65}$$

For $k = 2$ and $m = 0.5$, $\theta_{\text{width}} = \pi/4$, as expected.

### The Output Voltage

It can be shown that the average output voltage over a period of $T_s$ is given by

$$V_{o,\text{ave}} = m_a V_{\text{dc}} \qquad 0 < m_a \leq 1 \tag{9.66}$$

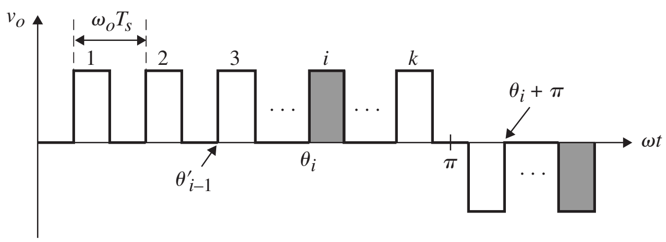



**Figure 9.31** $k$-pulse inverter output in a half-cycle.

The rms value for the *i*th pulse is given by

$$V_{o,\text{rms}} = \sqrt{\frac{1}{2\pi}\int_{\theta_i}^{\theta_1+\theta_{\text{width}}} 2kV_{\text{dc}}^2 \, d\omega t} = V_{\text{dc}}\sqrt{\frac{k}{\pi}\theta_{\text{width}}} \tag{9.67}$$

Recall that from Eq. (9.65) we have $\theta_{\text{width}} = \pi(m_a/k)$; hence, Eq. (9.67) becomes

$$V_{o,\text{rms}} = V_{\text{dc}}\sqrt{\frac{k}{\pi}\frac{\pi m_a}{k}} = V_{\text{dc}}\sqrt{m_a} \tag{9.68}$$

The rms of the output voltage is a function of the modulation index.

### Harmonics of *k* Equal Pulses

Let us determine the harmonic components for the *i*th pulse shown in Fig. 9.32. Since it is an odd function, only the odd harmonics exist in the $b_n$ coefficients. The harmonics of the output voltage due to the *i*th pulse acting alone is given by

$$V_{on,i} = \frac{V_{\text{dc}}}{\pi}\left[\int_{\theta_i}^{\theta_i+\theta_{\text{width}}} \cos n\omega t \, d\omega t - \int_{\theta_i+\pi}^{\theta_i+\theta_{\text{width}}+\pi} \cos n\omega t \, d\omega t\right]$$
$$= \frac{2V_{\text{dc}}}{n\pi}\sin\left(\frac{n\theta_{\text{width}}}{2}\right)\left[\sin\left(n\left(\theta_i+\frac{\theta_{\text{width}}}{2}\right)\right) - \sin\left(n\left(\pi+\theta_i+\frac{\theta_{\text{width}}}{2}\right)\right)\right] \tag{9.69}$$

For the total *k* pulses, $V_{on}$ is given by

$$V_{on} = \sum_{i=1}^{k}\frac{2V_{\text{dc}}}{n\pi}\sin\left(\frac{n\theta_{\text{width}}}{2}\right)\left[\sin\left(n\left(\theta_i+\frac{\theta_{\text{width}}}{2}\right)\right) - \sin\left(n\left(\pi+\theta_i+\frac{\theta_{\text{width}}}{2}\right)\right)\right]$$
$$\text{for } n = 1, 3, 5, \ldots \tag{9.70}$$

In terms of *k*, $m_a$, and $\theta_{\text{width}}$, using Eqs. (9.64) and (9.65), $\theta_i$ can be expressed as

$$\theta_i = \frac{\pi}{k}\left(i-\frac{1}{2}\right) - \frac{\theta_{\text{width}}}{2} \quad \text{where } i = 1, 2, \ldots, k \tag{9.71}$$

where

$$\theta_{\text{width}} = \frac{\pi}{k}m_a \tag{9.72}$$

It can be shown that the *n*th harmonic component for the *k*-pulse output voltage may be expressed as follows:

$$V_{on} = \frac{4V_{\text{dc}}}{n\pi}\sin\left(\frac{m_a\pi}{2k}n\right)\sum_{i=1}^{k}\sin\left(\frac{\pi}{k}(i-1/2)n\right) \tag{9.73}$$

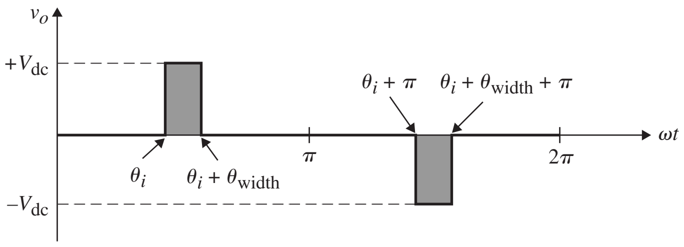



**Figure 9.32** Symmetrical representation of *i*th pulse for a given inverter output.

For illustration purposes, let us consider the example with $m = 0.5$ and the first nine harmonics, and then evaluate the total harmonic distortion as a function of the number of pulses.

To verify the above equation for a single pulse, we calculate the harmonic components for $k = 1$, $i = 1$, and $\theta_i = \alpha$, to yield

$$V_n = \frac{4V_{dc}}{n\pi}\sin\left(\frac{n\pi}{2}m_a\right)\sum_{i=1}^{1}\sin\left(\frac{\pi}{k}(i-1/2)n\right) \tag{9.74}$$

Substituting for $m_a$ from Eq. (9.72), Eq. (9.74) becomes

$$V_n = \left(\frac{4V_{dc}}{n\pi}\right)\sin\left(n\frac{\theta_{width}}{2}\right) \quad \text{for odd } n \tag{9.75}$$

Equation (9.75) represents the harmonic components of the output voltage for $k = 1$ as a function of the pulse width $\theta_{width}$. This equation is similar to what we derived previously as a function of $\alpha$.

## EXAMPLE 9.8

For uniform PWM with a value of $k = 5$ and a modulation index $m_a = 0.2$, calculate the output harmonic components up to the fifteenth harmonic.

***SOLUTION*** Using Eq. (9.73), Table 9.1 shows the values of the first 15 harmonics. Figure 9.33 shows the plot for the harmonic contents of Table 9.1. Figure 9.34 shows the harmonic ratios with respect to the fundamentals for $m = 0.2$ and $k = 1$ to $k = 7$ pulses per half-cycle.

From Table 9.1 for $m_a = 0.2$, we observe that as the number of pulses increases per half-cycle, the magnitude of the lower harmonics (third, fifth, seventh) decreases with respect to the fundamental component. Furthermore, there is an increase in magnitude for the higher-order harmonics with respect to the fundamental; however, such higher-order harmonics produce a negligible ripple that can be easily filtered out. Still, the ratio of the harmonic to the fundamental is relatively unchanged as the number of pulses increases within a half-cycle. The THD for this example is 223%! The higher the modulation index, the lower the THD.

## EXERCISE 9.8

Consider two equal pulses placed at $\omega t = \theta_i$ and $\omega t = \theta_i + \pi$, respectively, each with a width of $\theta_{width}$ as shown in Fig. 9.32. Show that the $n$th harmonic component is given by

$$v_{on} = V_n \sin n\omega t$$

**Table 9.1** Normalized Harmonics for $k = 5$ and $m_a = 0.2$

| Magnitude harmonic coefficient | $V_n/V_{dc}$ |
|---|---|
| $V_1$ | 0.258715 |
| $V_3$ | 0.098301 |
| $V_5$ | 0.078691 |
| $V_7$ | 0.095728 |
| $V_9$ | 0.245304 |
| $V_{11}$ | –0.238761 |
| $V_{13}$ | –0.088251 |
| $V_{15}$ | –0.068671 |

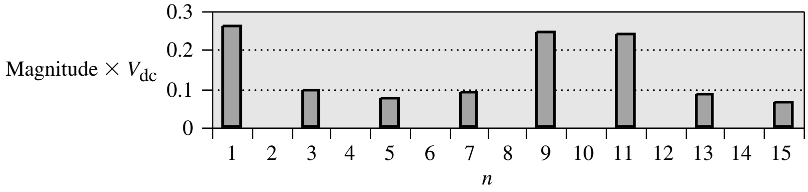



**Figure 9.33** Harmonic contents for $k = 5$ and $m_a = 0.2$.

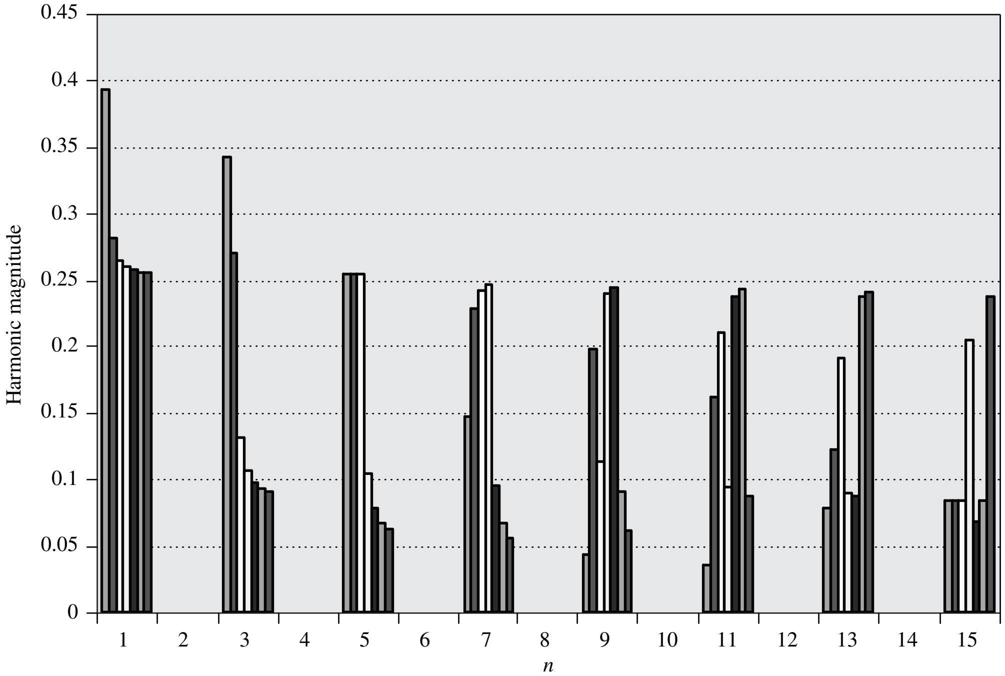



**Figure 9.34** Harmonic contents for $k = 1$ to $k = 7$, and $m_a = 0.2$.

where

$$V_n = \frac{2V_{dc}}{n\pi}\sin\left(\frac{n\theta_{width}}{2}\right)\left[\sin n\left(\theta_i + \frac{\theta_{width}}{2}\right) - \sin n\left(\pi + \theta_i + \frac{\theta_{width}}{2}\right)\right]$$

### 9.5.2 Sinusoidal PWM

**Basic Concept**

To illustrate the process of sinusoidal PWM, we refer to the simplified buck converter shown in Fig. 9.35. Recall that in PWM dc-dc converters, the duty cycle is modulated

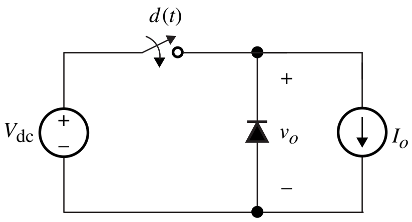



**Figure 9.35** Simplified buck converter.

between 0 and 1 to regulate the dc output voltage. In the steady-state, the duty cycle in PWM switch-mode converters is relatively constant and does not vary with time:

$$V_o = DV_{\text{dc}} \tag{9.76}$$

where $D$ is the duty cycle, representing the ratio of the *on* time of the switch to the switching period, and $V_o$ is the average output voltage.

If the duty cycle, $d(t)$, varies or is modulated according to a certain time function, with a modulating frequency, $f_o$, then it is possible to shape the output voltage waveform, $v_o$, in such a way that its average value over the modulating period synthesizes a sinusoidal waveform. For example, if the duty cycle is defined according to the function

$$d(t) = D_{\text{dc}} + D_{\max}\sin\omega_o t \tag{9.77}$$

where

$D_{\text{dc}}$ = dc duty cycle when no modulation exists
$D_{\max}$ = maximum modulation constant
$\omega_o$ = frequency of modulation

then the output voltage, $v_o$, is given by

$$\begin{aligned} v_o &= d(t)V_{\text{dc}} \\ &= V_{\text{dc}}D_{\text{dc}} + V_{\text{dc}}D_{\max}\sin\omega_o t \end{aligned} \tag{9.78}$$

For a buck converter, since the output voltage cannot be negative, then $D_{\max} \le D_{\text{dc}}$, as shown in Fig. 9.36.

As an example, if $D_{\text{dc}} = 0.5$, $D_{\max} = 0.8D_{\text{dc}}$, and $f_o = f_s/12$, the duty cycle is given by

$$d(t) = 0.5 + 0.4\sin(2\pi f_o t_i)$$

where $t_i = 0, 1, 2, 3, \ldots, 12$.

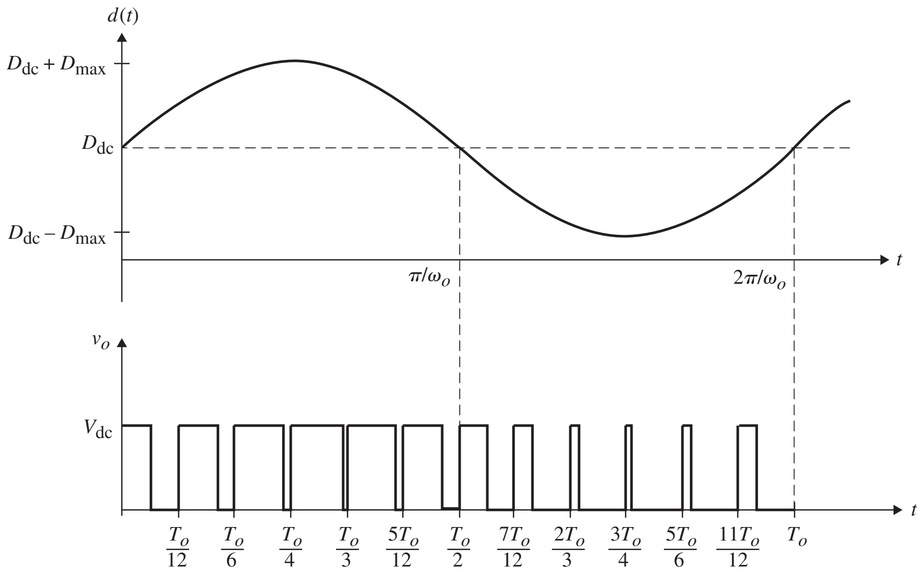



**Figure 9.36** Example of $D_{\text{dc}} = 0.5$, $D_{\max} = 0.8D_{\text{dc}}$, and $f_o = f_s/12$.

**Table 9.2** Example of 12-Pulse Sinusoidal PWM Signal

| Pulse | Time, $t_i$ | $d(t) = 0.5 + 0.4\sin(2\pi f_o t_i)$ |
|---|---|---|
| 1 | 0 | 0.50 |
| 2 | $T_o/12$ | 0.70 |
| 3 | $T_o/6$ | 0.85 |
| 4 | $T_o/4$ | 0.90 |
| 5 | $T_o/3$ | 0.85 |
| 6 | $5T_o/12$ | 0.70 |
| 7 | $T_o/2$ | 0.50 |
| 8 | $7T_o/12$ | 0.30 |
| 9 | $2T_o/3$ | 0.15 |
| 10 | $9T_o/12$ | 0.10 |
| 11 | $5T_o/6$ | 0.15 |
| 12 | $11T_o/12$ | 0.30 |

Since the switching frequency is 12 times faster than the modulating frequency, $f_o$, then $d(t)$ is sampled 12 times between $0 \le t < T_o$. This is illustrated in Table 9.2 for $0 \le t < T_o/2$ and six pulses in a half-cycle.

The preceding will now be applied to sinusoidal PWM inverters.

### PWM Modulation Function

Since the waveform is symmetrical around the $T_o/2$ point, we define $m(t)$ only in the first half-cycle. In the single-pulse inverter with a two-state output of Fig. 9.13(*b*), the modulation function is simply unity.

$$v_o = m(t)V_{\mathrm{dc}} \qquad m(t) = 1 \qquad 0 \le t < T/2 \tag{9.79}$$

In the single-pulse inverter with a tri-state output of Fig. 9.18(*a*), the modulation function in a half-cycle is given by

$$v_o = m(t)V_{\mathrm{dc}} \qquad m(t) = \begin{cases} 0 & 0 \le t < \alpha/\omega \\ +1 & \alpha/\omega \le t < T_o/2 - \alpha/\omega \\ 0 & T_o/2 - \alpha/\omega \le t < T_o/2 \end{cases} \tag{9.80}$$

In a *k*-pulse inverter with a constant duration, the modulation function is given by

$$v_o = m(t)V_{\mathrm{dc}} \qquad m(t) = \begin{cases} +1 & 0 \le t < d(t) \\ 0 & d(t) \le t < T_s \end{cases} \tag{9.81}$$

In the sinusoidal PWM technique, the pulse durations are adjusted to change slowly to follow the sinusoidal function. The modulation function is normally limited between 0 and 1. Since it is desired to have an output voltage with zero dc, the modulation function must be symmetrical around zero.

For the sinusoid PWM waveforms, we define as follows:

$$m(t) = M_{\max}\cos\omega_o t \tag{9.82}$$

where $w_o$ is the modulation frequency, which must be less than the switching frequency; i.e., $\omega_o << \omega_s$. $M_{\max}$ is the gain of the modulation function, which varies between 0 and 1. Normally the switching frequency is in the range of a few hertz to a 100 kHz while the modulation frequency is less than 500 Hz. As stated before, using the PWM process, we can extract a low-frequency signal from a train of high-frequency squarewaves. The higher the switching frequency of the squarewave output with respect to the desired low-frequency output signal, the more the output waveform approximates a sinusoidal.

A modulation gain of unity represents the largest possible output voltage, whereas a modulation gain of zero means the output waveform frequency equals the switching frequency, and the output voltage is the smallest.

### Switching Schemes

Depending on the switching sequence, the output voltage in PWM inverters can be either bipolar or unipolar. Figure 9.37 shows a bipolar output voltage in a PWM inverter. When the reference sinusoidal signal is larger or smaller than the triangular wave, the output equals $+V_{dc}$ or $-V_{dc}$, respectively. Both the half-bridge and full-bridge configurations are used in practice to generate a PWM output voltage waveform.

In bipolar voltage switching, $m_f$ is an odd number that is the same as the switching frequency, $f_s$. The output frequency, $f_o$, in unipolar voltage switching is twice the frequency in bipolar voltage switching ($m_f$ is doubled).

If both positive and negative sinusoidal control signals are available, then the switching sequence will produce a unipolar output waveform as shown in Fig. 9.38. The output waveforms for $v_{o1}$ and $v_{o2}$ are shown in Fig. 9.38($a$) and ($b$), respectively, and Fig. 9.38($c$) shows $v_o = v_{o1} - v_{o2}$. Here the triangular signal is chosen to be a sawtooth function.

### Signal Generation

Advanced digital and analog techniques exist in today's inverters to generate the driving signals that produce sinusoidal PWM. For illustration purposes, Fig. 9.39 shows a comparator that compares a triangular signal to a sinusoidal reference signal.

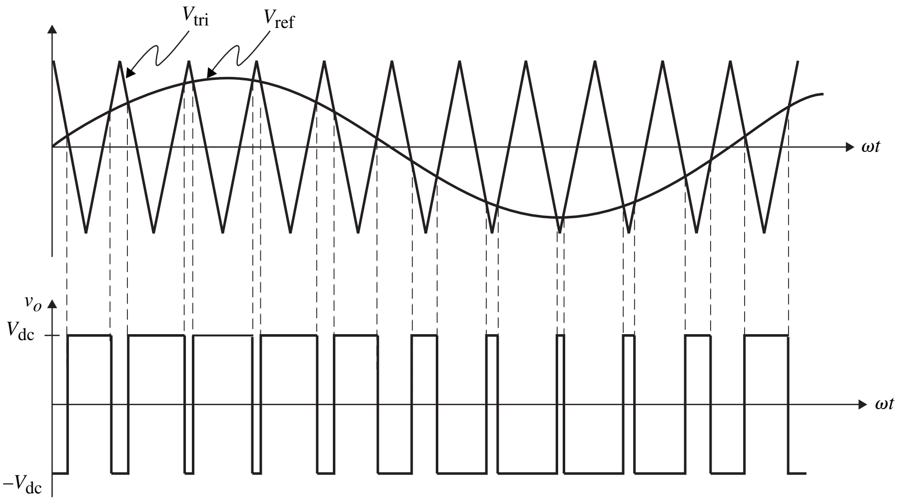



**Figure 9.37** Example of a bipolar PWM output waveform.

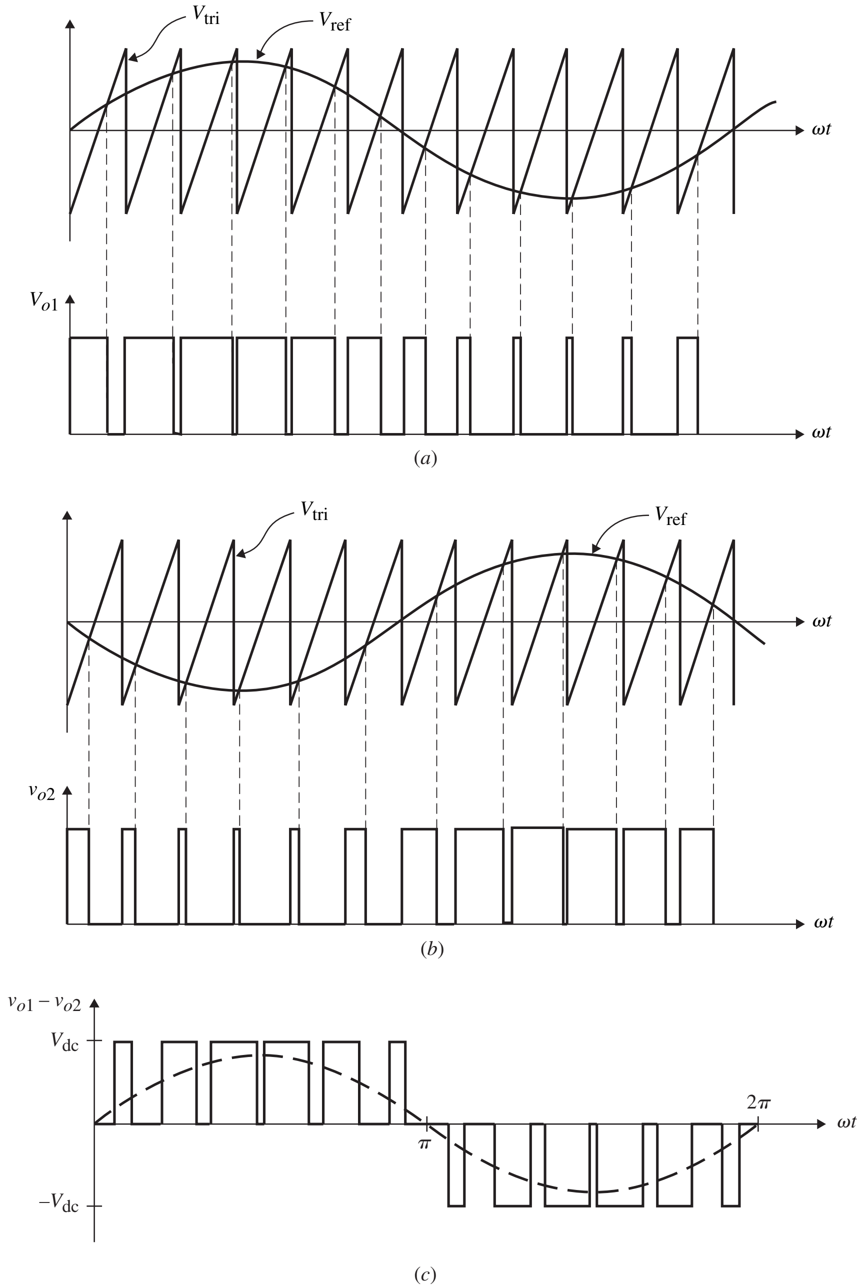



**Figure 9.38** Unipolar PWM output. (*a*) A positive sinusoidal reference to produce $v_{o1}$. (*b*) Positive sinusoidal reference to produce $v_{o2}$. (*c*) The differential output $v_o = v_{o1} - v_{o2}$ .

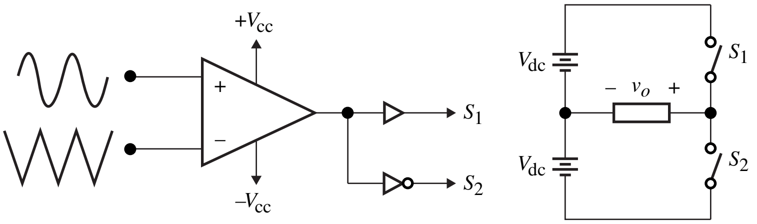



**Figure 9.39** Simplified circuit showing how signals are generated in sinusoidal PWM inverters.

### Analysis of Sinusoidal PWM

As stated before, the width of each pulse is varied in proportion to the instantaneous integrated value of the required fundamental component at the time of its event. In other words, the pulse width becomes a sinusoidal function of the angular position. In sinusoidal PWM, the lower-order harmonics of the modulated voltage waveform are highly reduced compared with the use of the uniform pulse-width modulation.

The output voltage signal in sinusoidal PWM can be obtained by comparing a control signal, $v_{\text{cont}}$, against a sinusoidal reference signal, $v_{\text{ref}}$, at the desired frequency. At the first half of the output period, the output voltage takes a positive value ($+V_{\text{dc}}$) whenever the reference signal is greater than the control signal. In the second half of the output period, the output voltage takes a negative value ($-V_{\text{dc}}$) whenever the reference signal is less than the control signal.

Similar to the case of equal-pulse PWM, the control frequency $f_{\text{cont}}$ (equal to the switching frequency) determines the number of pulses per half-cycle of the output voltage signal. Also, the output frequency $f_o$ is determined by the reference frequency $f_{\text{ref}}$. The amplitude modulation index, $m_a$, is defined as the ratio between the sinusoidal magnitude and the control signal magnitude.

$$m_a = \frac{V_{P,\text{ref}}}{V_{P,\text{cont}}} \tag{9.83}$$

The duration of the pulses is proportional to the corresponding value of the sinewave at the corresponding position. Then the ratio of any pulse duration to its corresponding time duration is constant, as shown in Figs. 9.40 and 9.42.

$$\frac{\beta_1}{y_1} = \frac{\beta_2}{y_2} = \frac{\beta_3}{y_3} \Rightarrow \frac{\beta_i}{y_i} = \text{constant} \tag{9.84}$$

The most important simplifying assumption here is that if the control frequency signal is very high with respect to the reference frequency signal, ($m_f >> 1$), then the value of the reference signal between two consecutive intersections with the control signal is almost constant. This is illustrated in Fig. 9.41.

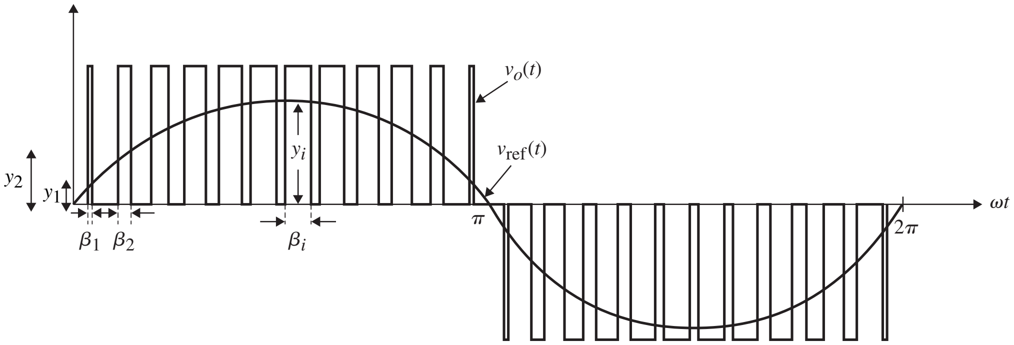



**Figure 9.40** PWM figure illustrating the constant ratio between the width and height of a given pulse.

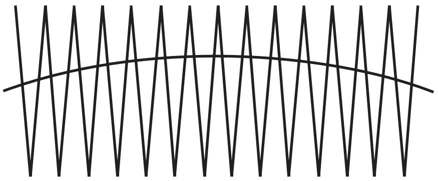


**Figure 9.41** High-frequency sinusoidal PWM.

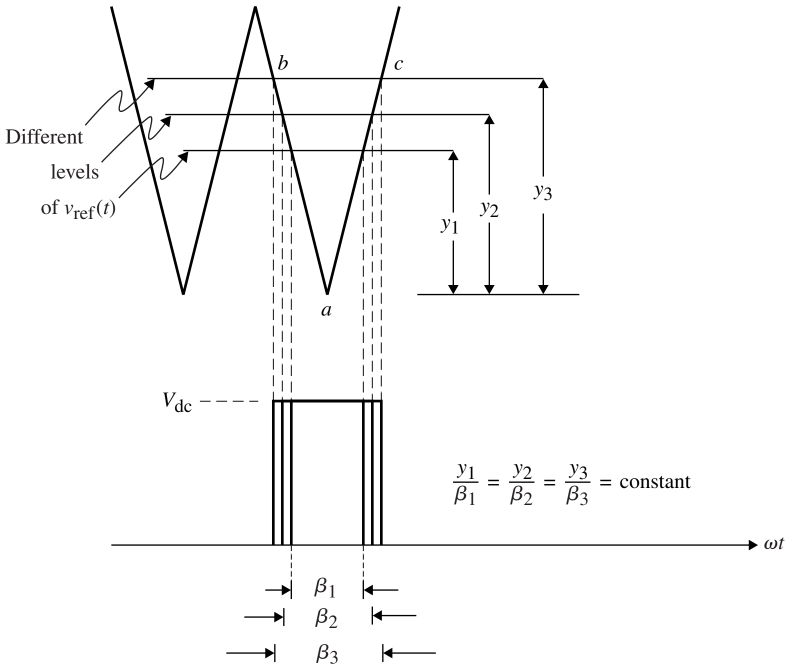



**Figure 9.42** Triangular approximation.

The proportional variation of each pulse width with respect to the corresponding sinewave amplitude can be seen by applying a triangular relationship, as shown in Fig. 9.42.

### Output Voltage Harmonics

The calculation of the sinusoidal PWM output voltage is the same as that of the uniform PWM output voltage. However, for sinusoidal PWM, the width of each pulse varies according to its position. The expression for the output voltage is obtained using a Fourier series transformation for $v_o$, given by

$$v_o(t) = \sum_{n=1,2,\ldots}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \tag{9.85}$$

Since the inverter output voltage is an odd function, only odd harmonics exist.

The calculation of the output voltage harmonic components can be done using a single pair of pulses as shown in Fig. 9.43.

$$\begin{aligned} V_{n,i} &= \frac{1}{\pi}\int_0^{2\pi} v_o(\omega t)\sin(n\omega t)\, d(\omega t) \\ V_{n,i} &= \frac{1}{\pi}\left[\int_{\theta_i}^{\theta_i+\theta_{wi}} V_{dc}\sin(n\omega t)\, d(\omega t) + \int_{\pi+\theta_i}^{\pi+\theta_i+\theta_{wi}} -V_{dc}\sin(n\omega t)\, d(\omega t)\right] \end{aligned} \tag{9.86}$$

Using the trigonometric relationship

$$\cos x - \cos y = -\left(2\sin\frac{x+y}{2}\right)\left(\sin\frac{x-y}{2}\right)$$

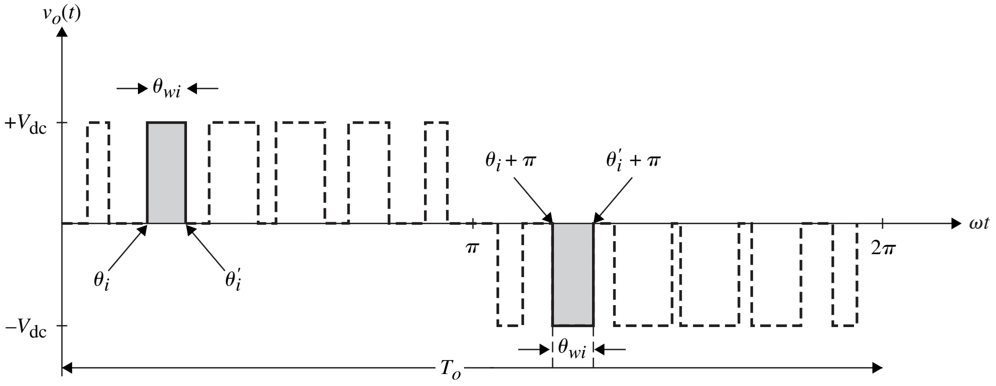



**Figure 9.43** Single sinusoidal PWM pair of pulses.

it can be shown that the harmonic component for a single pair of pulses is given by

$$V_{n,i} = \left(\frac{2V_{dc}}{n\pi}\right)\sin\left(n\frac{\theta_{wi}}{2}\right)\left[\sin n\left(\theta_i + \frac{\theta_{wi}}{2}\right) - \sin n\left(\theta_i + \frac{\theta_{wi}}{2} + \pi\right)\right] \tag{9.87}$$

Adding the contribution from all other pulses, the $i$th component of $v_o$ is given by

$$V_n = \sum_{i=1}^{k}\left(\frac{2V_{dc}}{n\pi}\right)\sin\left(n\frac{\theta_{wi}}{2}\right)\left\{\sin n\left(\theta_i + \frac{\theta_{wi}}{2}\right) - \sin n\left(\theta_i + \frac{\theta_{wi}}{2} + \pi\right)\right\} \tag{9.88}$$

where $\theta_i$ is the starting angle of the $i$th pulse and $\theta_{wi}$ is the pulse width at the corresponding angular position. Next, we estimate the width of each pulse, $\theta_{wi}$, for each $i$th pulse.

### Approximating the Pulse Width, $\theta_{wi}$

Assume each pulse is located at the discrete value of $\theta_i$, which represents the first intersection for the generation of the $i$th pulse. Then the approximated mathematical relation for the width is found using a geometrical relation as shown in Fig. 9.44.

From the geometry of the triangle *ABC* in Fig. 9.44, we have

$$h_y = V_{P,\text{tri}}, \qquad h_x = V_{P,\text{ref}}\sin\theta_i, \qquad y = \frac{T_o}{2k}\omega_o = \frac{\pi}{k}$$

The approximated width of the $i$th pulse, $\theta_{wi,\text{app}}$, is given by

$$\theta_{wi,\text{app}} = x \tag{9.89}$$

Since $\alpha = \beta$, then

$$\frac{h_x}{x} = \frac{h_y}{y} \tag{9.90}$$

Substituting for $h_y$, $h_x$, $y$, and $x$ in Eq. (9.90) and using $m_a = V_{P,\text{ref}}/V_{P,\text{tri}}$, Eq. (9.89) becomes

$$\theta_{wi,\text{app}} = \left(\frac{\pi}{k}\right)m_a\sin\theta_i \tag{9.91}$$

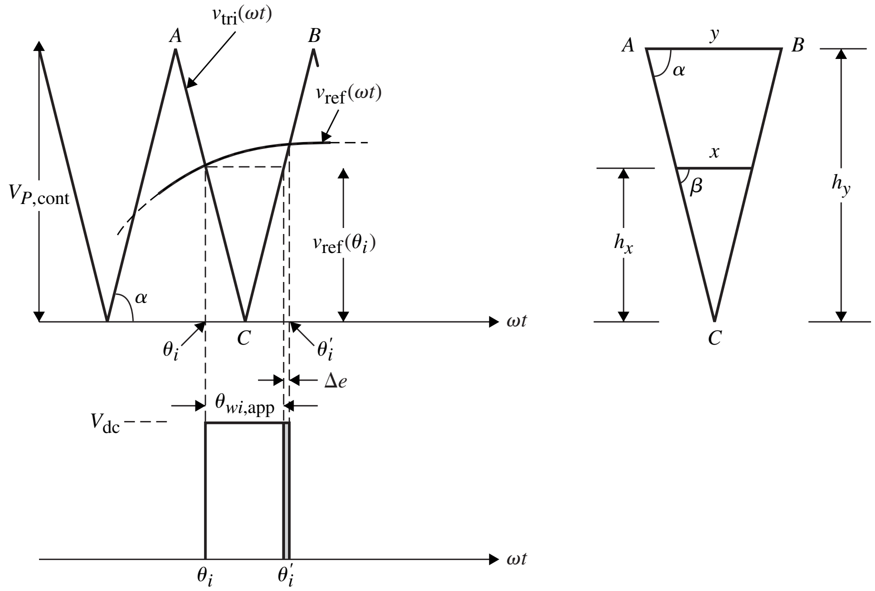



**Figure 9.44** Approximated pulse width, i.e., $\Delta e \approx 0$.

It can be seen that the exact width is produced for the consecutive intersections of the control signal and the sinusoidal signal at positions $\theta_i$ and $\theta'_i$. Therefore, an error is originated when the signal is considered constant and the value of the width is found using the approximated procedure. To reduce the width approximation error, it is necessary to increase the carrier frequency. Increasing the index frequency $m_f$ produces a considerable reduction in the difference of the reference signal values evaluated at $\theta_i$ and $\theta'_i$ respectively.

### Exact Expression for $\theta_{wi}$

For the *i*th pulse with an angle $\theta_i$, it can be easily shown that the point of intersection between the control signal and the $\omega t$ axis is $((2i-1)\pi/2k, 0)$, as shown in Fig. 9.45.

The general expression for $v_{\text{cont}}(\omega t)$ having all lines of negative slope is given by

$$v_{\text{cont}}(\omega t) = -\frac{m_f}{\pi} V_{P,\text{tri}}\left(\omega t - (2i-1)\frac{\pi}{m_f}\right) \quad i = 0, 1, 2, \ldots, k \tag{9.92}$$

The expression for the reference signal is given by

$$v_{\text{ref}}(\omega t) = V_{P,\,\text{ref}} \sin \omega t \tag{9.93}$$

Evaluating Eqs. (9.92) and (9.93) at $\omega t = \theta_i$, $v_{\text{cont}}(\theta_i) = v_{\text{ref}}(\theta_i)$, yields

$$-\frac{m_f}{\pi} V_{P,\,\text{tri}}\left(\theta_i - (2i-1)\frac{\pi}{m_f}\right) = V_{P,\,\text{ref}} \sin \theta_i \tag{9.94}$$

Equation (9.94) can be rewritten as

$$m_a \sin \theta_i = -\frac{m_f}{\pi}\theta_i + (2i-1) \tag{9.95}$$

The value of $\theta_i$ can be found by solving Eq. (9.95) numerically.

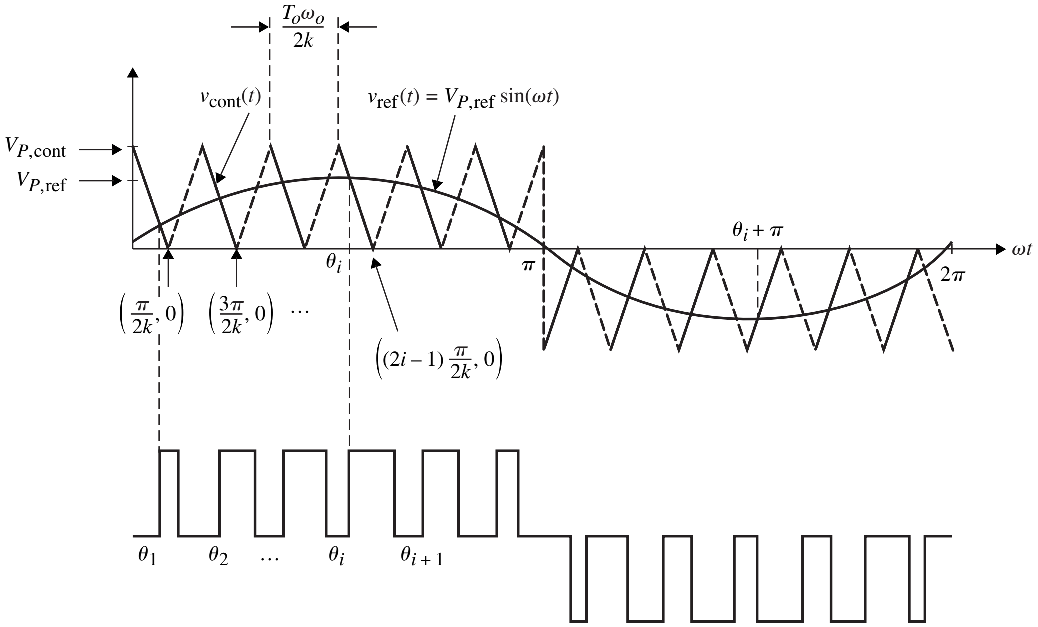



**Figure 9.45** Intersections between $v_{ref}$ and $v_{cont}$.

Consider the cases for different numbers of pulses per half-cycle shown in Fig. 9.46. The exact width for each pulse is found by adding the approximated width and the corresponding error $\Delta e$ in each interval. As shown in Fig. 9.46, for the two-pulse case we have $\Delta e_1 = \Delta e_2$ by symmetry, as well as for the cases of three and four pulses.

Generalizing for any symmetric pair of pulses, the calculation for any width can be done as shown in Fig. 9.47. From the triangle *ABC*

$$\tan\alpha = \frac{v_{ref}(\theta'_i) - v_{ref}(\theta_i)}{\Delta e_i} \tag{9.96}$$

From the triangle *DEF*

$$\tan\alpha = \frac{V_{P,tri}}{\left(\frac{T_o\omega_o}{4k}\right)} = \frac{V_{P,tri}}{\frac{1}{2}\frac{\pi}{k}} = \frac{2k}{\pi}V_{P,tri} \tag{9.97}$$

Equating Eqs. (9.96) and (9.97), $\Delta e_i$ may be expressed by

$$\Delta e_i = \left(\frac{\pi}{2k}\right)\left(\frac{v_{ref}(\theta'_i) - v_{ref}(\theta_i)}{V_{P,\,tri}}\right) \tag{9.98}$$

Substituting for $v_{ref}(\omega t) = V_{P,\,ref}\sin(\omega t)$, $k = \frac{1}{2}m_f$, and $m_a = V_{P,\,ref}/V_{P,\,cont}$ in Eq. (9.98), $\Delta e_i$ becomes

$$\Delta e_i = \frac{\pi m_a}{2k}(\sin\theta'_i - \sin\theta_i) \tag{9.99}$$

From the supplementary angle relation $\sin x = \sin(\pi - x)$, Eq. (9.99) becomes

$$\Delta e_i = \left(\frac{\pi m_a}{2k}\right)(\sin\theta_{k+1-i} - \sin\theta_i)$$

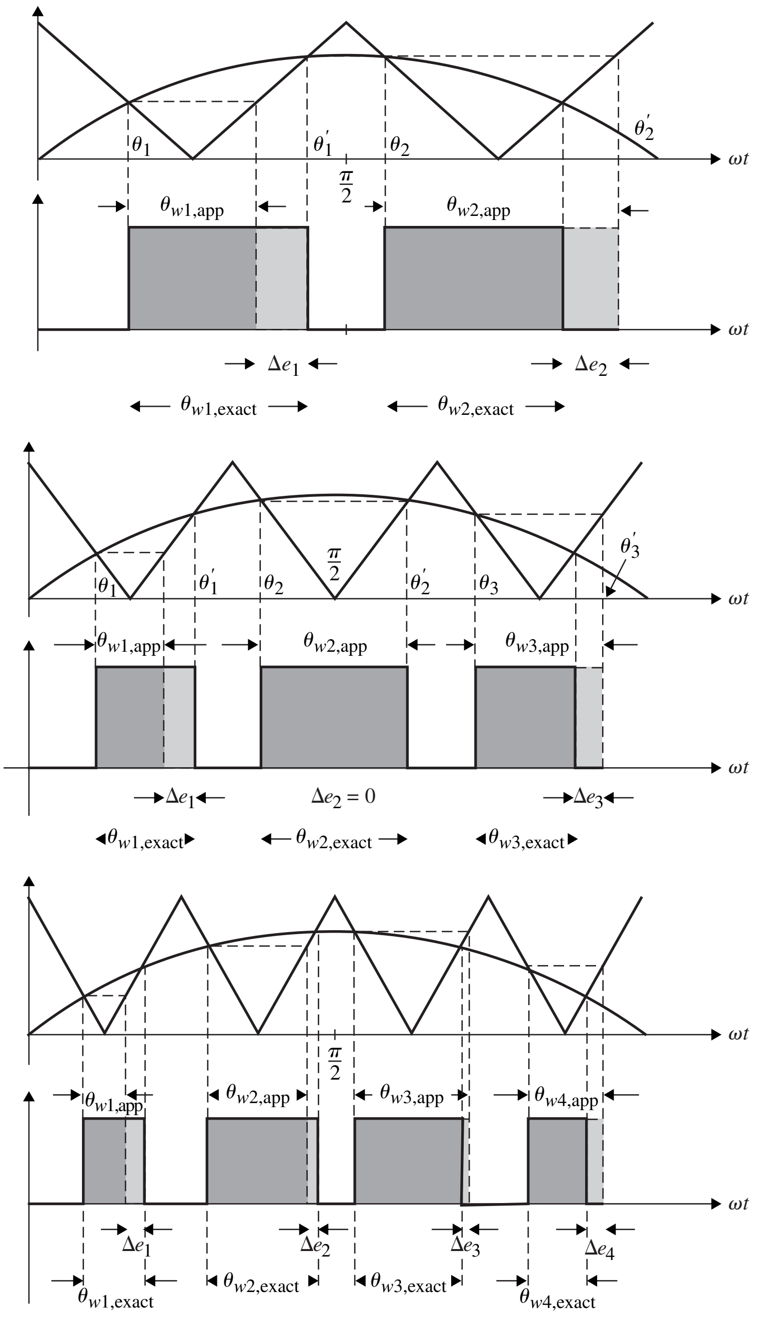



**Figure 9.46** Illustration of two, three, and four PWM pulses per half-cycle.

and since $\theta_{wi} = \theta_{wi,\text{app}} \pm \Delta e_i$, the general expression for the exact width is given by

$$\theta_{wi} = \frac{\pi m_a}{k}\left(\sin\theta_i \pm \frac{1}{2}\left(\sin\theta_i - \sin\theta_{k+i-1}\right)\right) \tag{9.100}$$

An increase in the control frequency causes a decrease in the value of $\Delta e$ and $\theta_{wi,\text{app}}$. Therefore, when the control frequency tends to infinity, the width of $\Delta e$ tends to zero. It can be shown that $\Delta e$ decreases at a higher rate than does $\theta_{wi,\text{app}}$.

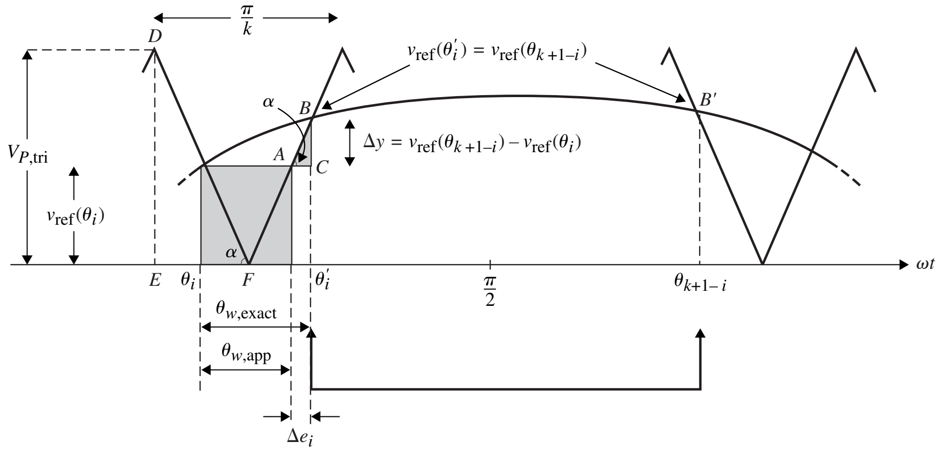



**Figure 9.47** Symmetric pair of pulses about $\pi/2$.

## EXAMPLE 9.9

Consider a full-bridge inverter with an $R-L$ load with a sinusoidal PWM output voltage with a reference frequency $f_o$ of 60 Hz, $V_{dc} = 280$ V, $m_a = 0.6$, and $m_f = 24$.

**(a)** Find the carrier frequency (triangular wave).

**(b)** Find the number of pulses per half-cycle.

**(c)** Find the angles of intersection and the pulse widths in a half-cycle.

**(d)** Find the harmonic components.

**(e)** Find the total harmonic distortion.

***SOLUTION***

**(a)** $f_s = m_f \times f_o = 24 \times 60 \text{ Hz} = 1.44 \text{ kHz}$

**(b)** $k = \frac{24}{2} = 12$ pulses

**(c)** The exact angles were numerically calculated using Mathcad as shown in the following table.

| Starting angle $\theta_i$ (degrees) | Arriving angle $\theta_i$ (degrees) | Pulse width $\theta_{wi}$ (degrees) |
|---|---|---|
| 6.955 | 8.137 | 1.182 |
| 20.895 | 24.364 | 3.469 |
| 34.926 | 40.419 | 5.493 |
| 49.099 | 56.242 | 7.143 |
| 63.474 | 71.773 | 8.299 |
| 78.093 | 86.994 | 8.901 |
| 93.006 | 101.903 | 8.897 |
| 108.226 | 116.526 | 8.300 |
| 123.759 | 130.901 | 7.106 |
| 139.582 | 145.076 | 5.494 |
| 155.644 | 159.104 | 3.460 |
| 171.863 | 173.045 | 1.182 |

**(d)** Harmonic components, in volts:

| | | |
|---|---|---|
| $V_1 = 167.931$ | $V_{11} = -0.019$ | $V_{21} = 19.909$ |
| $V_3 = -0.048$ | $V_{13} = 0.085$ | $V_{23} = 103.541$ |
| $V_5 = 0.127$ | $V_{15} = -0.021$ | $V_{25} = -103.74$ |
| $V_7 = 0.03$ | $V_{17} = -0.071$ | $V_{27} = -19.736$ |
| $V_9 = -0.118$ | $V_{19} = 1.058$ | $V_{29} = -0.898$ |

These values are plotted in Fig. 9.48.

Using the approximated width, we obtain the following, using MATLAB:

| | | |
|---|---|---|
| $V_1 = 167.742$ | $V_{11} = 0.02$ | $V_{21} = 16.071$ |
| $V_3 = 0.769$ | $V_{13} = -0.021$ | $V_{23} = 108.129$ |
| $V_5 = 6.917 \times 10^{-3}$ | $V_{15} = 8.579 \times 10^{-3}$ | $V_{25} = -99.036$ |
| $V_7 = 7.869 \times 10^{-3}$ | $V_{17} = -4.769 \times 10^{-3}$ | $V_{27} = -23.464$ |
| $V_9 = -6.94 \times 10^{-3}$ | $V_{19} = 0.564$ | $V_{29} = -2.089$ |

These results are plotted in Fig. 9.49.

**(e)** The total harmonic distortion is approximated by

$$\text{THD} \approx \sqrt{\frac{1}{V_1^2}(V_{21}^2 + V_{23}^2 + V_{25}^2 + V_{27}^2 + V_{29}^2)} = 0.89$$

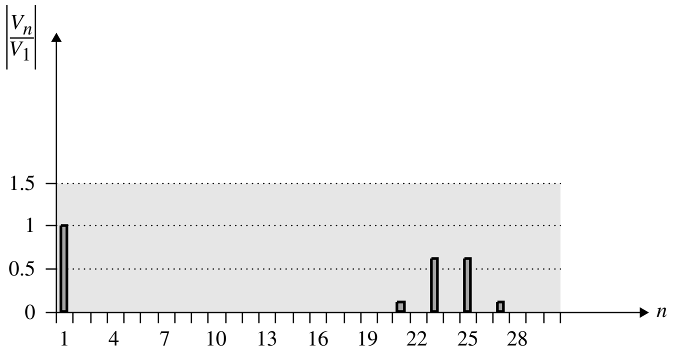



**Figure 9.48** First 29 harmonics using exact analysis for Example 9.9.

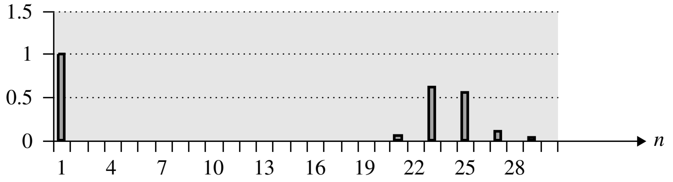



**Figure 9.49** A spectrum plot for the first 29 harmonics using the approximate analysis.

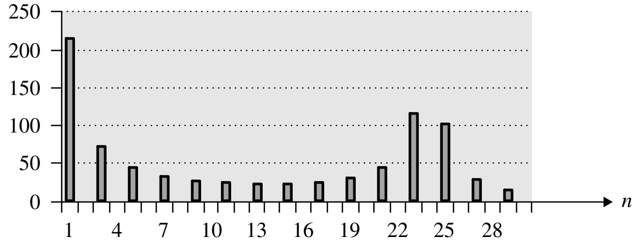



**Figure 9.50** Uniform PWM for Example 9.9.

If we compare the uniform PWM approach with the sinusoidal PWM approach, we can notice a considerable reduction in the harmonic components. The next graph, Fig. 9.50, shows a plot of the normalized magnitude of the harmonic components using uniform pulse-width modulation with a modulation index of 0.6 for 12 pulses per half-period.

---

As we know, the inverter will produce a rectangular pulse waveform. To transform the rectangular waveform generated by the switching process to an output waveform that resembles a sinusoid wave, we need to use a harmonic filter. A low-pass filter can be used due to the high attenuation property without affecting the fundamental harmonic.

However, a perfect low-pass filter is impractical to realize, and the lower harmonics that are close to the fundamental will affect our desired output. Therefore, we need to maximally reduce the lower-order harmonics so the filter will allow the first harmonic to pass in our variable-frequency output.

## 9.6 THREE-PHASE INVERTERS

Consider the three-phase full-bridge dc-ac inverter shown in Fig. 9.51. To obtain a set of balanced line-to-line output voltages, the switching sequence of switches $S_1$-$S_6$ should produce a sequence of pulses whose summation at any given time is zero. As a result, it can be shown that in a one-pulse phase voltage, the conduction angle is $\pi/3$. The switch numbering follows the sequence of switching. The bidirectional switch implementation of $S_1$-$S_6$ allows an inductive-load current flow.

Figure 9.52($a$) and ($b$) shows two switching sequences for $S_1$-$S_6$ with each switch conducting for $\pi$ and $2\pi/3$, respectively. Both sequences produce a similar output voltage. To avoid shorting the voltage source $V_{dc}$, the switching sequence of $S_1$-$S_6$ must ensure that the $S_1$-$S_4$, $S_3$-$S_6$, and $S_5$-$S_2$ pairs are not switched on at the same time.

Figure 9.53($a$) shows the circuit configuration for a three-phase inverter with a wye load and splitting input capacitors. The inverters generate three-phase output voltages. The switches are switched in such a way that the voltages $v_b$ and $v_c$ are shifted by $2\pi/3$. Figure 9.53($b$) shows the three-phase output voltages $v_a$, $v_b$, and $v_c$. Figure 9.53($c$) shows the three-phase line-to-line voltages $v_{ab}$, $v_{bc}$, and $v_{ca}$.

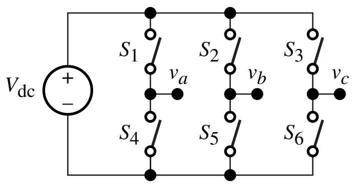



**Figure 9.51** Three-phase inverter.

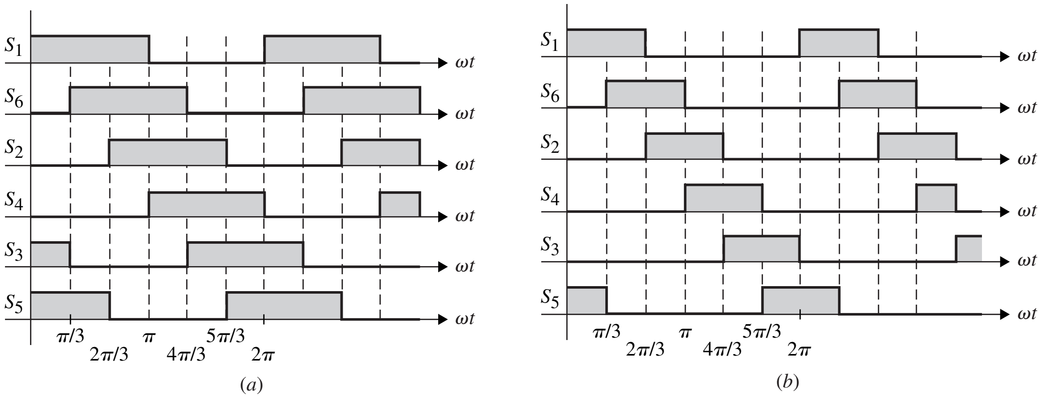



**Figure 9.52** Switching sequence. (*a*) Conduction equals $\pi$. (*b*) Conduction equal $2\pi/3$.

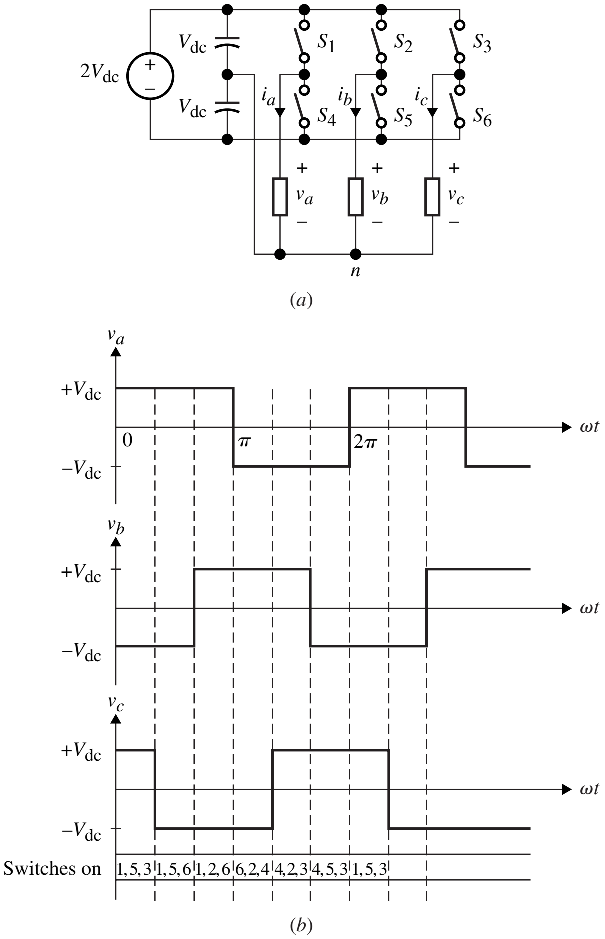
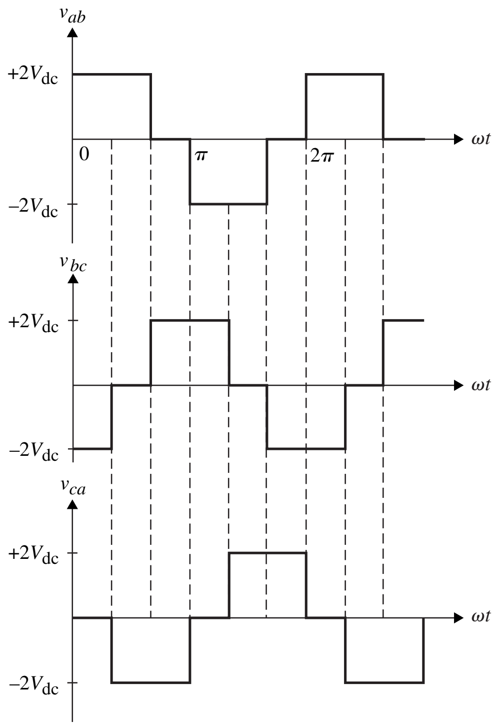



**Figure 9.53** (*a*) Three-phase inverter with wye load. (*b*) Phase voltage waveforms using switching sequence of Fig. 9.52(*a*).

Figure 9.54 shows another line-to-line voltage using the switching sequence of Fig. 9.52(*a*). Three-phase inverters are also implemented by employing three single-phase inverters as shown in Fig. 9.55. The transformers' primary windings must be isolated from one another, and the transformers' secondary windings may be connected in a delta or wye configuration.




(c)

**Figure 9.53** (*continued*) (*c*) Line-to-line voltages $v_{ab}$, $v_{bc}$, and $v_{ca}$.

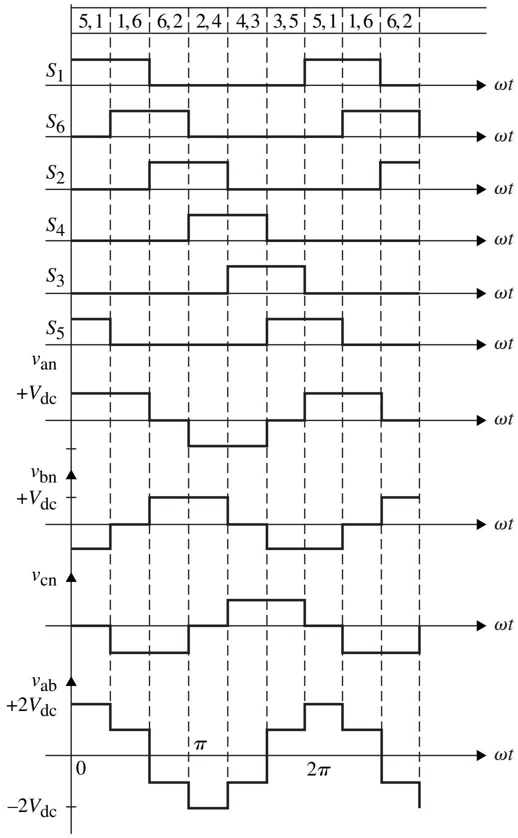



**Figure 9.54** Switching sequence for three-phase inverter to produce $v_{ab}$.

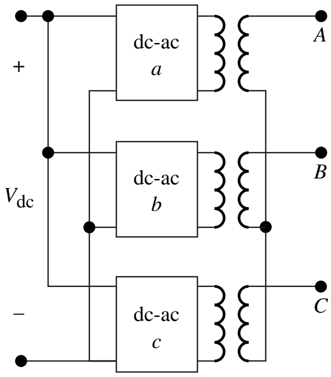



**Figure 9.55** Three-phase inverter using three single-phase dc-ac inverters.

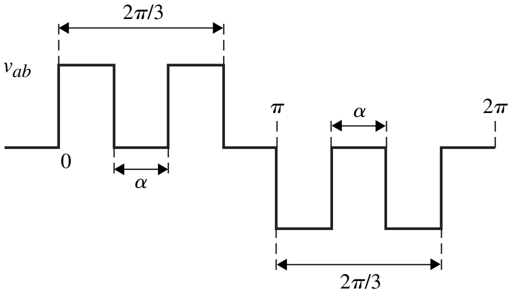



**Figure 9.56** $\alpha$ control in line-to-line phase voltage.

From Fourier analysis, $v_{ab}$ is given by

$$v_{ab}(t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_{dc}}{n\pi} \cos\frac{n\pi}{6} \sin n\left(\omega t + \frac{\pi}{6}\right)$$

$$v_{bc}(t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_{dc}}{n\pi} \cos\frac{n\pi}{6} \sin n\left(\omega t - \frac{\pi}{2}\right) \qquad (9.101)$$

$$v_{ab}(t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_{dc}}{n\pi} \cos\frac{n\pi}{6} \sin n\left(\omega t - \frac{7\pi}{6}\right)$$

Notice all triple harmonics ($n = 3, 6, 9, 12, \ldots$) are zero.

Similarly, the output voltage control can be implemented using three-phase inverters. Figure 9.56 shows one possible three-phase output under $\alpha$ control.

## EXAMPLE 9.10

Consider the three-phase inverter circuit shown in Fig. 9.57 under an inductive-resistive load. Sketch the voltage waveform for the phase voltages $v_{an}$, $v_{bn}$, and $v_{cn}$ and line-to-line voltages $v_{ab}$, $v_{bc}$, and $v_{ca}$ using the switching sequence shown in Fig. 9.58($a$).

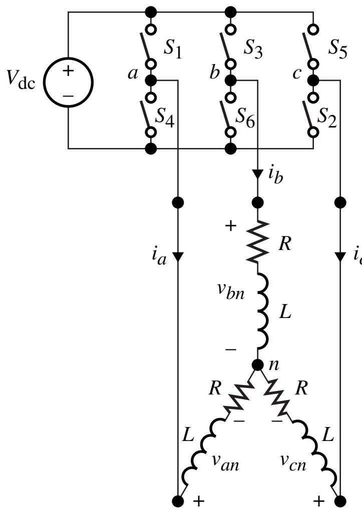



**Figure 9.57** Three-phase inverter under an inductive-resistive load.

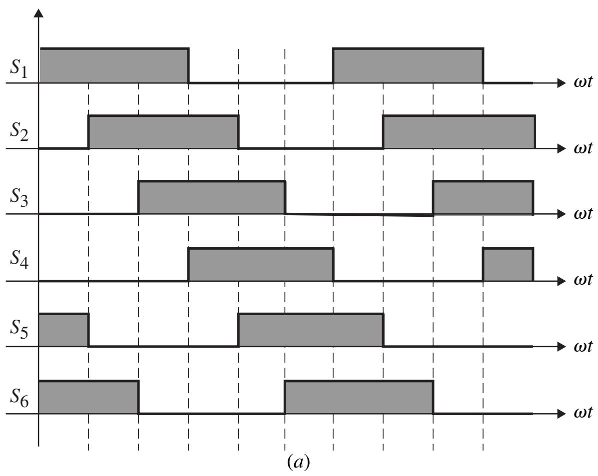



**Figure 9.58**
($a$) Switching sequence for Fig. 9.57.

***SOLUTION*** Figure 9.58($b$) shows the three phase voltages $v_{an}$, $v_{bn}$, and $v_{cn}$. Figure 9.59 shows the line-to-line voltages $v_{ab}$, $v_{bc}$, and $v_{ca}$.

***Mode 1***: $\mathbf{0 \le \omega t < \pi/3}$. $S_5$, $S_6$, and $S_1$ are on. $S_2$, $S_3$, and $S_4$ are off.

It can be shown that since the loads are the same, the pulse voltages are given by

$$v_{an} = \frac{1}{3}V_{dc}, \qquad v_{bn} = -\frac{2}{3}V_{dc}, \qquad v_{cn} = \frac{1}{3}V_{dc}$$

The equivalent circuit for this mode is shown in Fig. 9.60($a$).

The line-to-line voltages are given by

$$v_{ab} = V_{dc}, \qquad v_{bc} = -V_{dc}, \qquad v_{ca} = 0$$

***Mode 2***: $\mathbf{\pi/3 \le \omega t < 2\pi/3}$. $S_1$, $S_2$, and $S_6$ are on. $S_3$, $S_4$, and $S_5$ are off. The equivalent circuit for this mode is shown in Fig. 9.60($b$).

The line-to-line voltages are given by

$$v_{ab} = V_{dc}, \qquad v_{bc} = 0, \qquad v_{ca} = -V_{dc}$$

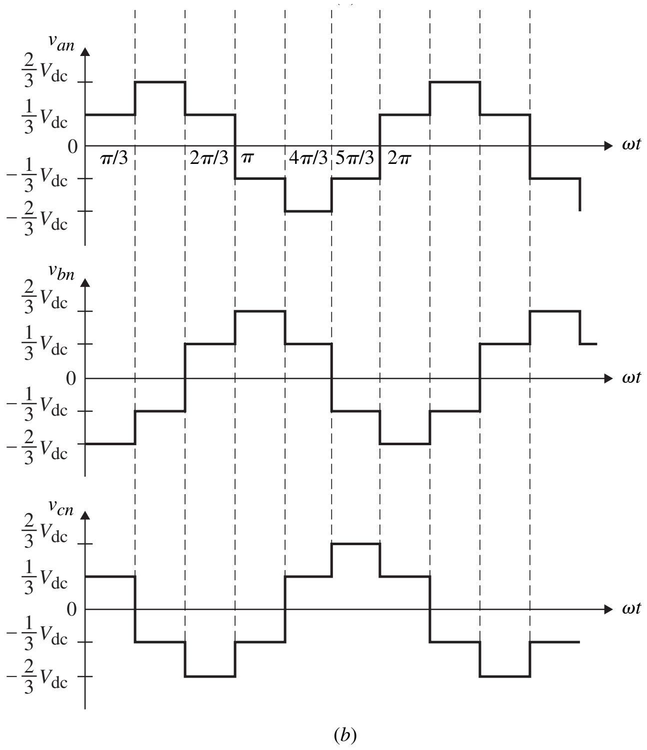



**Figure 9.58** (*continued*) (*b*) Phase voltages $v_{an}$, $v_{bn}$, and $v_{cn}$.

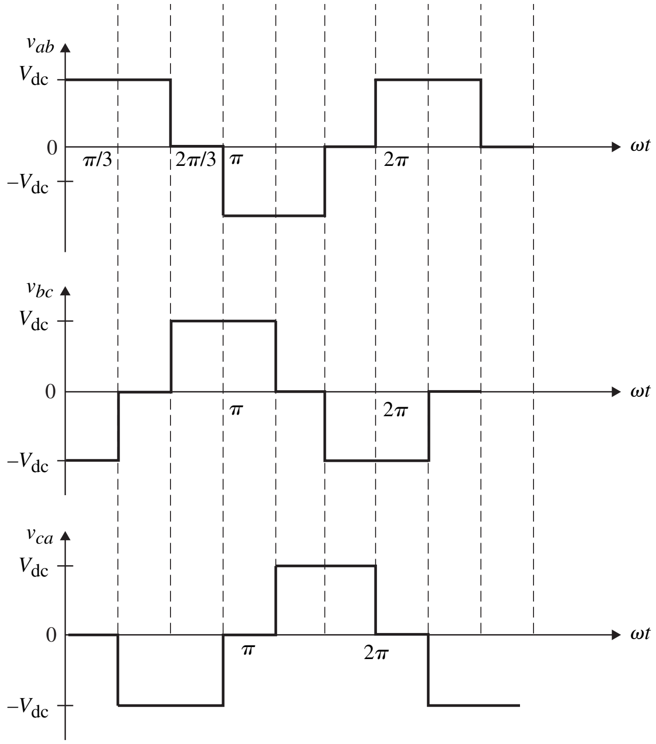



**Figure 9.59** Three-phase line-to-line voltages $v_{ab}$, $v_{bc}$, and $v_{ca}$.

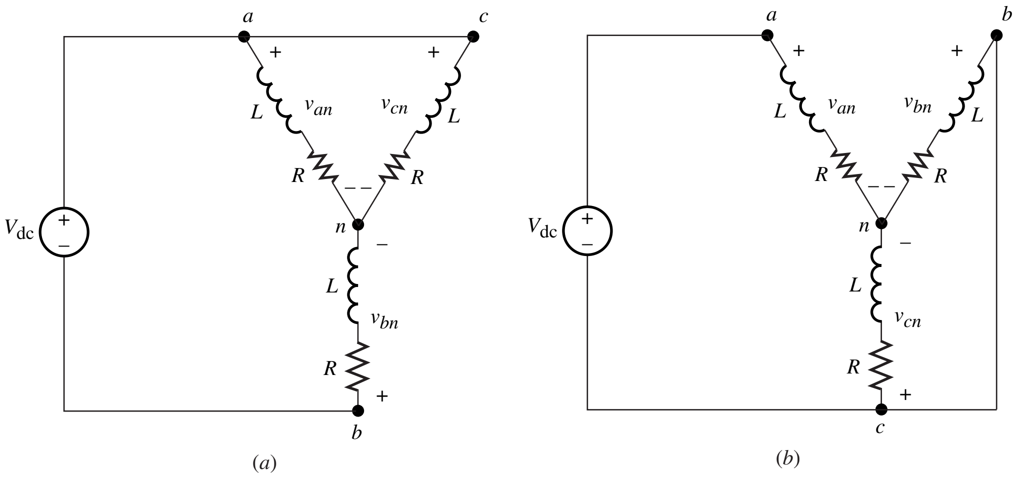



**Figure 9.60** (*a*) Equivalent circuit for mode 1. (*b*) Equivalent circuit for mode 2.

and the phase voltages are given by

$$v_{an} = \frac{2V_{dc}}{3}, \quad v_{bn} = -\frac{V_{dc}}{3}, \quad v_{cn} = \frac{-V_{dc}}{3}$$

It can be shown that the remaining modes will produce the waveforms in Fig. 9.58(*b*) and Fig. 9.59.

## EXERCISE 9.9

Sketch the load currents $i_{ab}$, $i_{bc}$, and $i_{ca}$ for the three-phase inverter shown in Fig. E9.9. Assume the device conduction sequence is as given in Fig. 9.58.

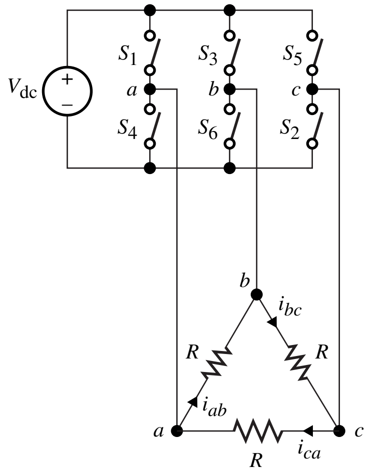



**Figure E9.9** Three-phase inverter for Exercise 9.9.

## EXERCISE 9.10

Show that the fundamental components for the load currents in Exercise 9.9 are given by

$$i_{ab,1} = \frac{4V_{dc}}{R\pi}\cos\frac{\pi}{6}\sin(\omega t + \pi/6)$$

$$i_{bc,1} = \frac{4V_{dc}}{R\pi}\cos\frac{\pi}{6}\sin(\omega t - \pi/2)$$

$$i_{ca,1} = \frac{4V_{dc}}{R\pi}\cos\frac{\pi}{6}\sin(\omega t - 7\pi/6)$$

## 9.7 CURRENT-SOURCE INVERTERS

In the current-source inverter, the dc input is a current source regardless of the input voltage variation. Practically, the dc current source is implemented using a large dc inductor in series with the dc voltage source as shown in Fig. 9.61. The dc supply is of a high impedance (because of the high input inductance). Since the dc inductance ($L_{dc}$) is large, $I_{dc}$ is nearly constant.

The output current waveform is determined by the circuit's topology and switching sequence, while the output voltage waveform is determined by the nature of the load. Loads with low impedance to the harmonics are normally used. Figure 9.62 shows a tall-bridge current-source inverter. The switches $S_1$-$S_4$ are implemented using GTOs and diodes; the SCR implementation is difficult in the current-source inverter since the turn-off time is hard to set.

As with the voltage-source inverter, depending on the switching sequence of $S_1$, $S_2$, $S_3$, and $S_4$, two possible output currents may be obtained, as shown in Fig. 9.63 (*a*) and (*b*). Figure 9.63(*a*) uses a 50% duty cycle, and the output current and its rms value are controlled by varying the value of $I_{dc}$. In Fig. 9.63(*b*) the duty cycle is less than

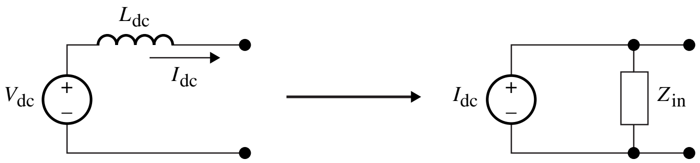



**Figure 9.61** Practical dc current source implementation.

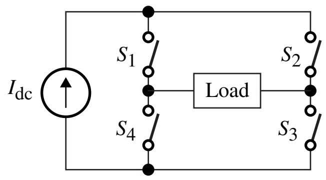



**Figure 9.62** Full-bridge current-source inverter.

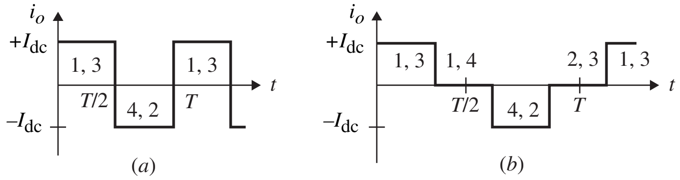



**Figure 9.63** Possible output current waveforms.

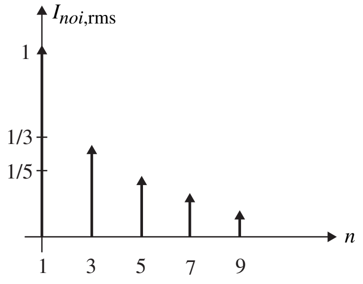



**Figure 9.64** Frequency spectra for the normalized rms current harmonics.

50%. There are three output states: $+I_{dc}$, 0, and $-I_{dc}$. In both cases, the output voltage waveforms will depend on the nature of the load.

From Fourier analysis, the load current is given by

$$i_o(t) = \frac{4I_{dc}}{\pi}\left[\sin\omega t + \frac{\sin 3\omega t}{3} + \cdots + \frac{\sin n\omega t}{n}\right] \qquad n = 1, 3, 5, 7, \ldots \tag{9.102}$$

The fundamental component of $i_o(t)$ is given by

$$i_{o1}(t) = \frac{4I_{dc}}{\pi}\sin\omega t \tag{9.103}$$

The peak and rms values of the fundamental component are

$$i_{o1,\text{peak}} = \frac{4I_{dc}}{\pi} \tag{9.104a}$$

$$i_{o1,\text{rms}} = \frac{2I_{dc}\sqrt{2}}{\pi} \tag{9.104b}$$

Let the normalized rms of the *i*th harmonic be given by

$$I_{no,\text{rms}} = \frac{I_{oi,\text{rms}}}{I_{o1,\text{rms}}} \qquad i = 1, 3, 5, \ldots \tag{9.105}$$

| Harmonics | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| $I_{no,\text{rms}}$ | 1 | 1/3 | 1/5 | 1/7 | 1/9 |

The frequency spectra for the normalized rms current harmonics are shown in Fig. 9.64.

Consider the ideal current-source inverter shown in Fig. 9.65($a$). Assume $v_o = V_o\sin(\omega t + \theta)$ and $S_1$-$S_4$ are switched according to the sequence shown in Fig. 9.65($b$), with a 50% switching duty cycle with no $\alpha$ control. By shifting $S_1$ to the left by $\alpha$ and $S_3$ to the right by $\alpha$, we allow an interval during which the input current source is shorted, as shown in Fig. 9.65($c$).

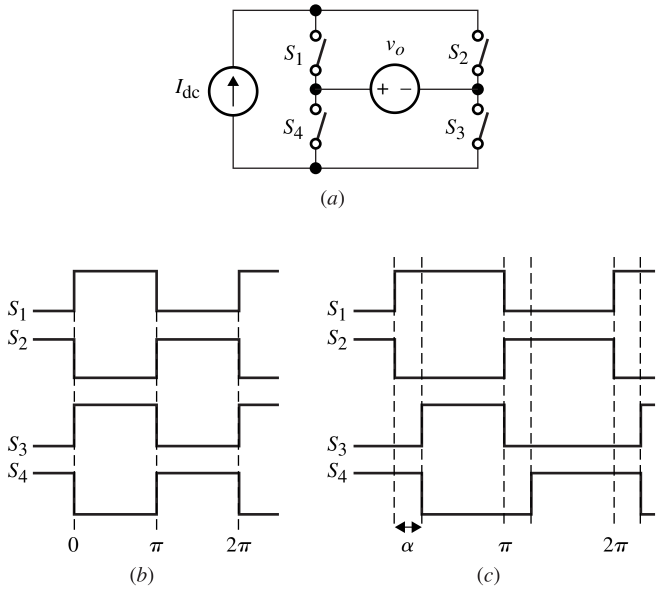



**Figure 9.65** (*a*) Ideal current-source inverter ($\alpha = 0$). (*b*) 50% switch ($\alpha = 0$). (*c*) Less than 50% switch ($\alpha \neq 0$).

## PROBLEMS

### *Half-Bridge and Full-Bridge Inverters*

**9.1** Derive Eq. (9.5), which gives the initial inductor current in a half-bridge inverter under a resistive-inductive load.

**9.2** Repeat Exercise 9.2 by including the first, third, and fifth harmonics.

**9.3** Consider the half-bridge inverter and its driven waveforms shown in Fig. P9.3 with the following inverter parameters: $V_{dc} = 185$ V, $R = 10\ \Omega$, and $f = 60$ Hz.

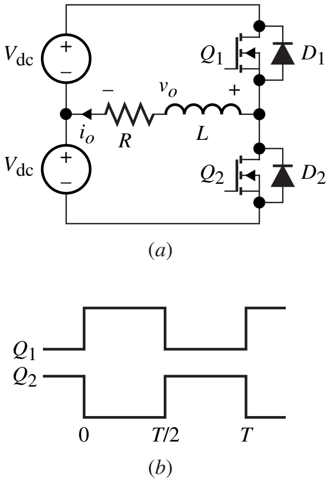



**Figure P9.3**

**(a)** Determine the average diode current for $L = 1$ mH, 5 mH, 10 mH, and 20 mH.

**(b)** As the load's time constant increases with respect to $T/2$, what is the effect on the load's harmonics?

**9.4** Consider the half-bridge inverter of Fig. 9.9 with the following circuit components: $V_{dc} = 408$ V, $R = 6\ \Omega$, and $f = 60$ Hz, and $L = 20$ mH.

**(a)** Derive the exact expression for $i_L$.

**(b)** Derive the expression for the fundamental component of $i_L(t)$.

**(c)** Determine the average diode and transistor currents.

**(d)** Determine the average power delivered to the load.

**9.5** Consider the full-bridge inverter shown in Fig. P9.5(*a*) with the given parameters. Assume the switching sequence of $S_1$-$S_4$ produces the output voltage waveform shown in Fig. P9.5(*b*).

**(a)** Determine the expression for $i_o(t)$ and sketch it.

**(b)** With $S_1$-$S_4$ replaced by a parallel combination of switch and diode, find the peak and average switch and diode currents.

**(c)** Find the average power delivered to the load.

**(d)** Determine the fundamental and third harmonic peak voltage. What is the percentage of the third harmonic with respect to the fundamental?

**(e)** Repeat part (*d*) for $L = 50$ mH, 100 mH, and 200 mH.

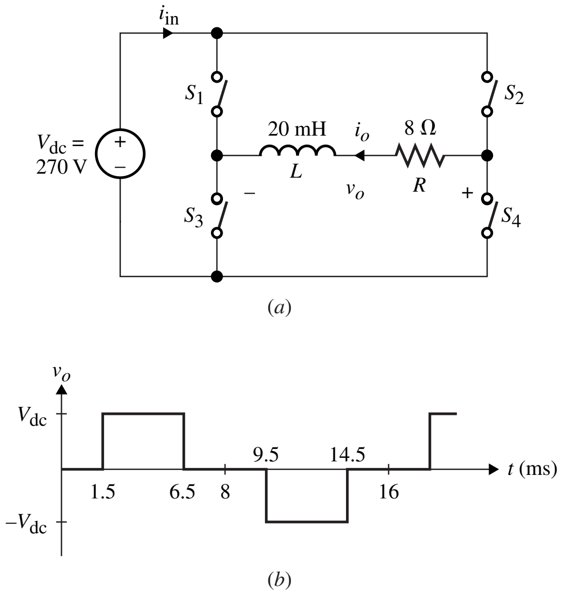



**Figure P9.5**

**9.6** Consider the ideal dc-to-ac inverter shown in Fig. P9.6. Assume $S_1$-$S_4$ and $S_2$-$S_3$ are switched alternatively at a 50% duty cycle with a switch period of *T*. Let the output current be given by $i_o(t) = I_P \sin \omega t$, where $\omega = 2\pi/T$.

**(a)** Derive the expression for the instantaneous power delivered to the load.

**(b)** Determine the average power delivered to the load.

**(c)** Discuss the requirement for a practical switch implementation for $S_1$-$S_4$.

**(d)** Repeat part (*c*) for $i_o(t) = I_P \sin(\omega t - \theta)$.

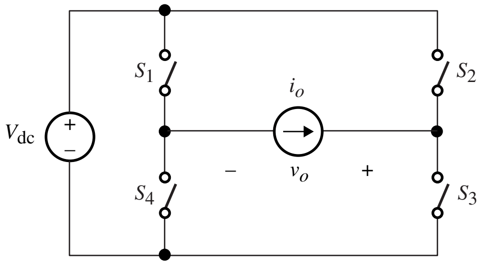



**Figure P9.6**

**9.7** Consider the full-bridge inverter circuit with an *RLC* load shown in Fig. P9.7(*a*). Use $L = 42$ mH, $R = 18\ \Omega$, $C = 900\ \mu$F, and $V_{dc} = 270$ V. Assume the switch sequence of $S_1$-$S_4$ produces the output voltage waveform shown in Fig. P9.7(*b*). Determine the peak fundamental component for the load current $i_o(t)$.

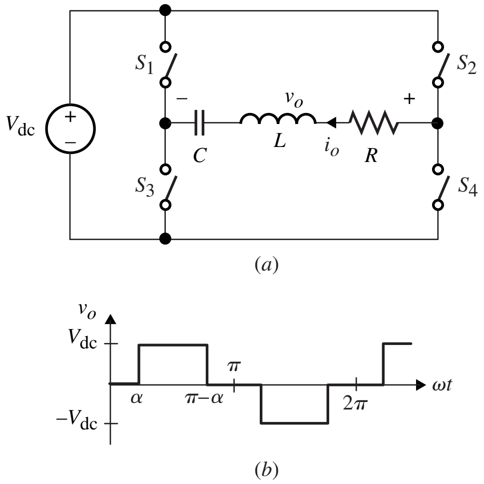



**Figure P9.7**

**9.8** *(Approximation analysis for an RL load)* Consider the half-bridge inverter of Fig. 9.8(*b*) with an *RL* load. Assume the load's time constant, $L/R$, is longer than half of the switching period, i.e., $L/R >> T/2$, so that the third and higher harmonics are neglected. Show that the average power delivered to the load is given by

$$P_{\text{ave}} = |S| \cos\theta$$

where $|S|$ is the magnitude of the reactive power, which is given by

$$|S| = \frac{8V_{\text{dc}}^2}{\pi^2|Z|}\cos^2\alpha$$

$$\theta = \tan^{-1}(\omega L/R)$$

and

$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

**9.9** Consider the switching *RL* circuit shown in Fig. P9.9(*a*) with $v_s$ given in Fig. P9.9(*b*).

**(a)** Derive the expression for $i_L(t)$ in terms of $V_{\text{dc}}$ and $\tau_n$, where $\tau_n = \tau/T = L/RT$.

**(b)** Give the expressions for $i_L(0)$, $i_L(t_1)$, and $i_L(T/2)$ in terms of $V_{\text{dc}}$ and $\tau_n$.

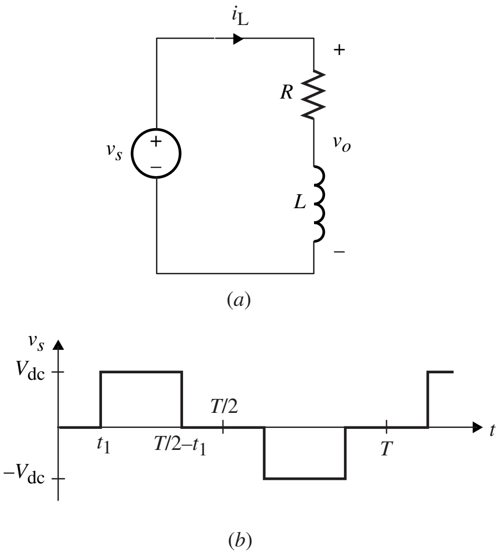



**Figure P9.9**

## *Harmonics*

**9.10** Show that the harmonics for the generalized squarewave with a pulse width equal to $\pi - 2\alpha$ and a delay angle $\alpha$ as shown in Fig. P9.10 is given by

$$v_o(t) = V_1\sin(\omega t) + V_3\sin(3\omega t) + V_5\sin 5(\omega t) + \cdots + V_n\sin(n\omega t)$$

where

$$V_n = \frac{-4}{n\pi}\cos n\alpha$$

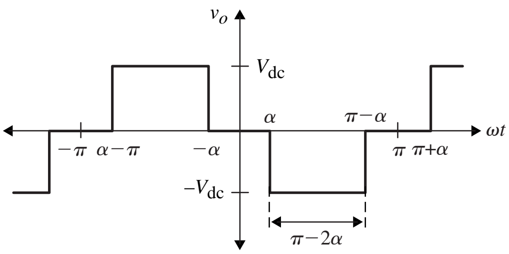



**Figure P9.10**

**9.11** Consider the output voltage of an inverter shown in Fig. P9.11 that uses $\alpha$ control to eliminate two harmonics from the output.

**(a)** For $\beta_1 = \pi/4$, $\beta_2 = \pi/3$, and $\beta_3 = 3\pi/4$ find the fundamental, third, and fifth harmonics.

**(b)** Redesign the problem for $\alpha_1$, $\alpha_2$, and $\alpha_3$ to eliminate the third and fifth harmonics.

**9.12** Consider an inverter circuit that produces the stepped output waveform shown in Fig. P9.12. It can be shown that by properly selecting $\alpha_1$ and $\alpha_2$, the third and fifth harmonics can be eliminated from the output voltage. Determine $\alpha_1$ and $\alpha_2$ to accomplish this.

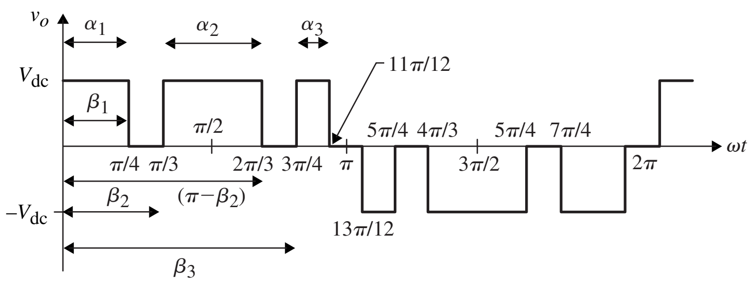



**Figure P9.11**

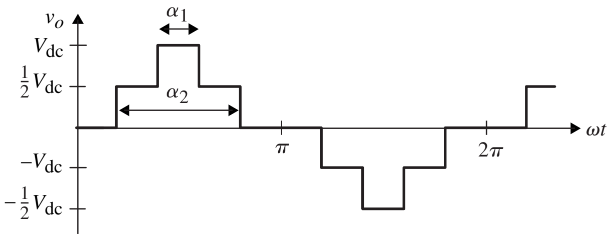



**Figure P9.12**

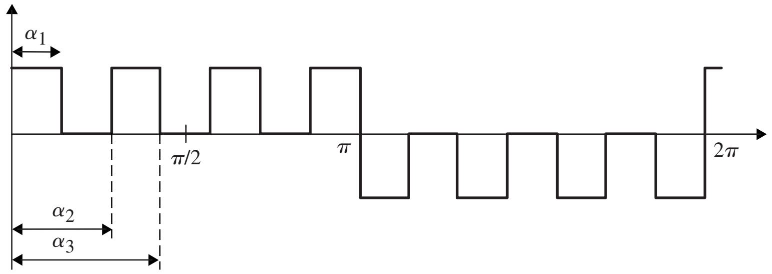



**Figure P9.14**

**9.13** Determine the THD for the stepped waveform shown in Problem 9.12 by using only up to the ninth harmonic.

**9.14** In a certain output voltage for an inverter, it is desired to eliminate the third, fifth, and seventh harmonics by controlling the switching sequence of the power devices. The following equation was obtained for the *n*th output voltage harmonic of Fig. P9.14.

$$v_{on} = \frac{4V_{dc}}{n\pi}(1 - \cos n\alpha_1 + \cos n\alpha_2 - \cos n\alpha_3)$$

Determine $\alpha_1$, $\alpha_2$, and $\alpha_3$ for a zero value of third, fifth, and seventh harmonics. (*Hint:* Use a numerical solution method or an iterative technique.)

***Uniform Pulse-Width Modulation***

**9.15** Derive the relation for $\theta_i$ given by Eq. (9.64),

$$\theta_i = \frac{\pi}{k}\left[i - \frac{m_a}{2} - \frac{1}{2}\right]$$

for the *i*th pulse in a *k*-pulse output.

**9.16** Derive the expression given in Eq. (9.65), which shows that the width of the pulse in a uniform PWM output is constant and is given by $\theta_{width} = \pi m_a / k$.

**9.17** Determine the rms value for a uniform PWM inverter output with $V_{dc} = 260$ V and $m_a = 0.8$. Determine the width of each pulse if the output has 24 pulses per cycle.

**9.18** Derive Eq. (9.75).

**9.19** Repeat Example 9.8 for $k = 10$ and $m_a = 0.4$. How does the THD compare to the THD obtained in that example?

***Sinusoidal Pulse-Width Modulation***

**9.20** Sketch the output voltage for the buck converter whose output is modulated according to Eq. (9.78). Assume $D_{dc} = D_{max} = 0.5$, $V_{dc} = 1$ V, and $f_o = f_s/10$.

**9.21** Derive Eq. (9.88).

**9.22** Derive the closed-form expression for $\theta_i$ given in Eq. (9.95).

**9.23** Determine the approximate width of the fourteenth pulse in a sinusoidal PWM inverter output of Fig. 9.44 with $k = 18$ and $m_a = 0.5$.

**9.24** Repeat Example 9.9 for $m_a = 1.0$.

***Three-Phase Inverters***

**9.25** The waveform shown in Fig. P9.25 represents one possible way to implement a line-to-line output voltage in a three-phase inverter control. By varying the angle $\alpha$, the rms value of the output can be controlled.

Show that the *n*th harmonic component for the line-to-line voltage is given by

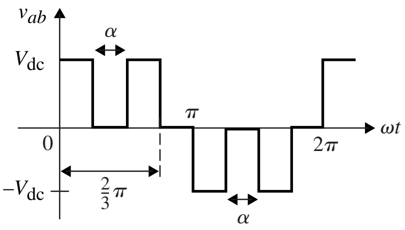



**Figure P9.25**

$$V_{ab,n} = \frac{4}{n\pi} V_{\text{in}}\left(\sin n\frac{\pi}{3} - \sin\frac{n\alpha}{2}\right)$$

$$\left(\cos n\left(\omega t - \frac{\pi}{3} - \frac{\alpha}{2}\right)\right)$$

and show that the total harmonic distortion is given by

$$\text{THD}_v^2 = \frac{\pi(2\pi/3-\alpha)}{8(\sin(\pi/3) - \sin(\alpha/2))^2} - 1$$

**9.26** Derive the harmonic components for $v_{ab}$, $v_{bc}$, and $v_{ca}$ in Fig. 9.53(*c*).

**9.27** One way to reduce the harmonics of a line-to-line voltage in a three-phase inverter is to produce an output waveform like the one shown in Fig. 9.54. Determine the *n*th harmonic component of $v_{ab}$.

**9.28** Derive the harmonic component for $v_{ab}$ of Fig. P9.28. (*Hint*: See Appendix B.)

**9.29** The equivalent circuit for a line-to-line voltage in a three-phase inverter is given in Fig. P9.29. **(a)** Show that in the steady state the inductor current at $\omega t = 0$ is given by

$$I_L(0) = -\frac{V_{\text{dc}}}{3R}\frac{(1 + e^{-1/6\tau_n} - e^{-1/3\tau_n} - e^{-1/2\tau_n})}{1 + e^{-1/2\tau_n}}$$

where

$$\tau_n = \frac{\tau}{T} = \frac{L}{RT}, \qquad T = \frac{2\pi}{\omega}$$

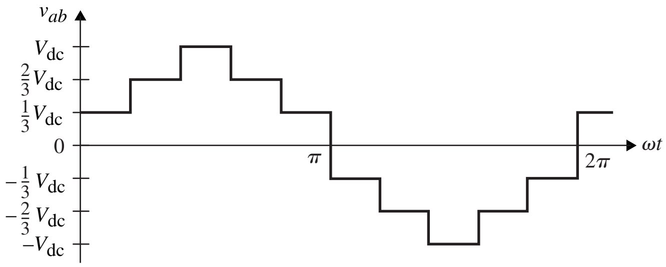



**Figure P9.28**

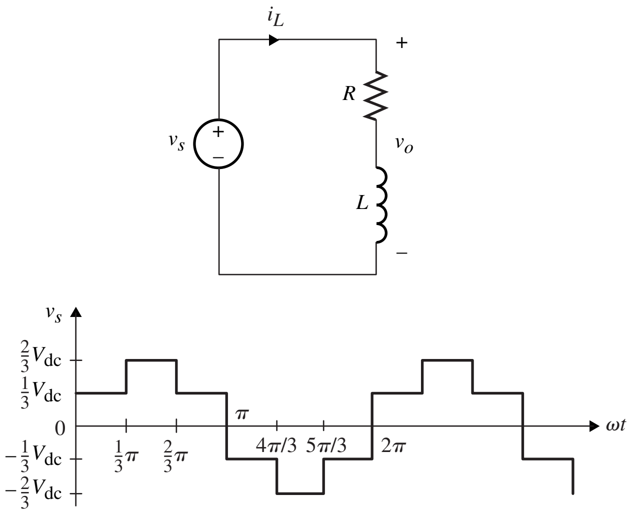



**Figure P9.29**

**(b)** Show that the steady-state inductor current value at $\omega t = \pi/3$ is given as a function of $I_L(0)$ as follows:

$$I_L(\pi/3) = \left(I_L(0) - \frac{V_{dc}}{3R}\right)e^{-1/6\tau_n} + \frac{V_{dc}}{3R}$$

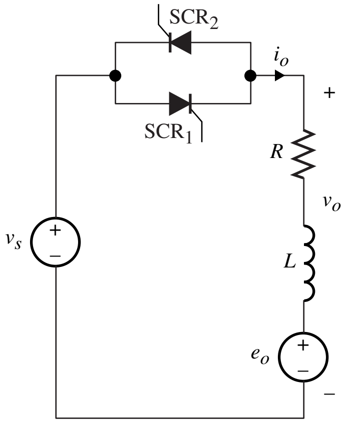



**Figure P9.30**

**(c)** Determine the time at which the flyback diode of $S_1$ stops conducting, assuming $L = 12$ mH, $R = 18\ \Omega$, $f = 400$ Hz, and $V_{dc} = 260$ V.

### *General Problems*

**9.30** Consider the single-phase ac controller shown in Fig. P9.30, with the induction motor load modeled by a resistor, inductor, and emf ac source.

Assume the emf voltage, $e_o(t)$, is given by

$$e_o(t) = V_o \sin(\omega t + \theta_o)$$

and

$$v_s(t) = V_s \sin(\omega t)$$

Here we assume that the source frequency and the emf voltage frequency are equal. Also assume $SCR_1$ and $SCR_2$ are triggered continuously at $\alpha_1 = 30°$ and $\alpha_2 = 330°$, respectively. Use $R = 0.85\ \Omega$, $L = 10.2$ mH, $V_s = 120$ V, $\omega = 2\pi(60)$, and $V_o = 85$ V.

**(a)** Sketch the waveform of $i_o(t)$ for the full cycle.

**(b)** Derive the expression of $i_o(t)$ for the first half-cycle.

**9.31** Figure P9.31(*a*) shows a single-phase power electronic circuit with an $R - L$ load known as an *ac controller*. These controllers are widely used in single- or three-phase arrangements for various residential and commercial applications such as heating, motor speed control, and power factor correction. A possible switch implementation is shown in Fig. P9.31(*b*) with back-to-back SCRs to produce bidirectional current flow and bidirectional voltage blocking.

Discuss the operation of the circuit that produces the output waveform shown in Fig. P9.31(*c*).

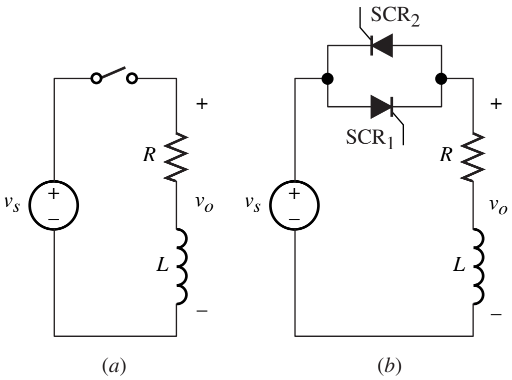



**Figure P9.31** (*a*) ac controller with inductive-resistive load. (*b*) SCR implementation.

**9.32 (a)** Show that if the exact expression for $i_L(t)$ is used in Fig. 9.9, then the fundamental component is given by

$$i_L(t) = I_{L1}\sin(\omega t + \theta)$$

where

$$I_{L1} = \frac{4V_{dc}}{R\pi}\sqrt{\frac{4\pi^2\tau_n^2(e^{-1/\tau_n} - e^{-1/2\tau_n}) + 4\pi^2\tau_n^2 + 1}{4\pi^2\tau_n^2 + 1}}$$

$$\theta = \tan^{-1}\left[2\pi\tau_n e^{-1/2\tau_n} - \frac{1}{2\pi\tau_n} - 2\pi\tau_n\right]e^{1/(2\tau_n)}$$

**(b)** Show that the simplified expression when $L/R >> T$ is given by

$$I_{L1} \approx \frac{4V_{dc}}{\pi}\frac{1}{\sqrt{R^2 + (\omega L)^2}}$$

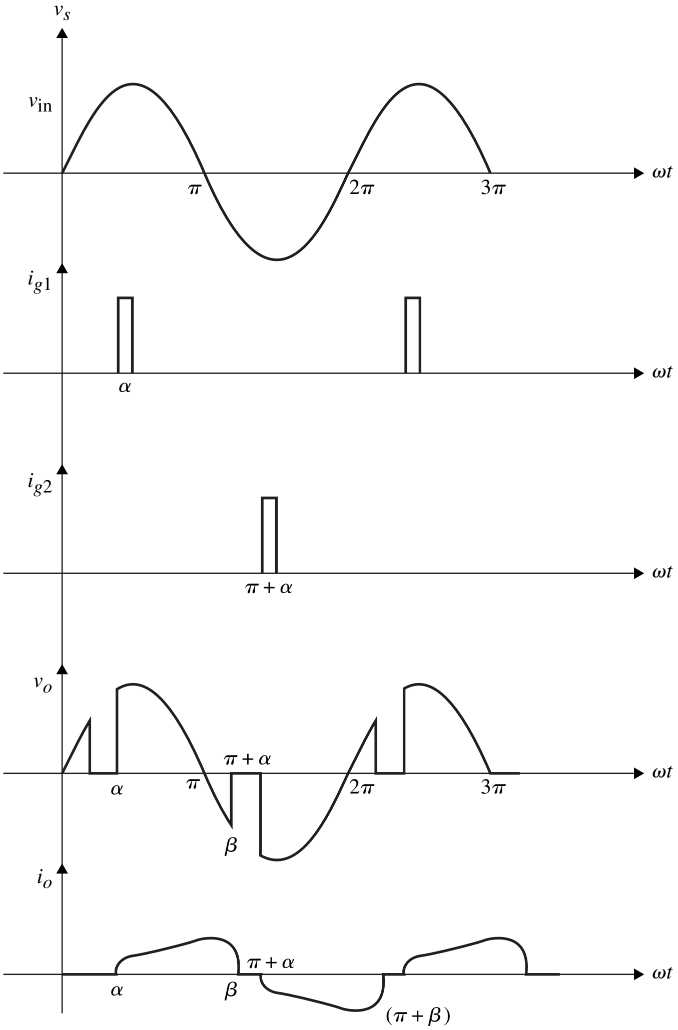



**Figure P9.31** (*continued*) (*c*) Output waveform.